12. Electromagnetic Induction

Can you recall?

1. What is the force experienced by a moving charge in a magnetic field?
2. What is the torque experienced by a current carrying loop kept in a magnetic field?
3. What is the magnetic dipole moment of a current carrying coil?
4. What is the flux of a vector field through a given area?

12.1 Introduction:

So far, we have focussed our attention on the generation of electric fields by stationary charges and magnetic fields by moving charges. During the early decades of nineteenth century, Oersted, Ampere and a few others established the fact that electricity and magnetism are inter-related. A question was then naturally raised whether the converse effect of – the moving electric charges produce magnetic fields – was possible? That is, can we produce electric current by moving magnets?

In 1831, Faraday in England performed a series of experiments in connection with the generation of electric current by means of magnetic flux. In the same year Joseph Henry (1799-1878) demonstrated that electric currents were indeed produced in closed circuits or coils when subjected to time-varying magnetic fields. The outcome of these experiments led to a very basic and important law of electromagnetism known as Faraday's law of induction. An electromotive force (emf) and, therefore, a current can be induced in various processes that involve a change in magnetic flux. The experimental observations of Faraday are summarized as given below:

i) When a magnet approaches a closed circuit consisting of a coil (Fig. 12.1), it produces a current in it. This current is known as induced current.

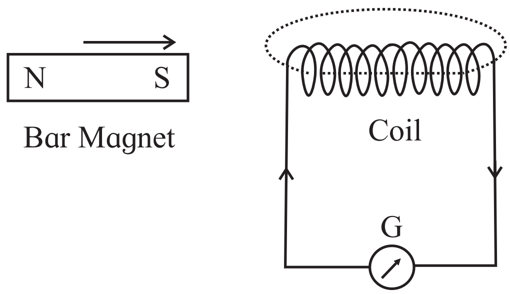


Fig. 12.1: A bar magnet approaching a closed circuit consisting of a coil and galvanometer (G).

ii) When the magnet is taken away from the closed circuit a current is again produced but in the opposite direction with respect to that in experiment (i).

iii) If instead of the magnet, the coil is moved towards the magnet or away from it, an induced current is produced in the coil (i.e., in the closed circuit).

iv) If the polarity of approaching or receding magnet is changed the direction of induced current in the coil is also changed.

v) The magnitude of induced current depends on the relative speed of the coil with respect to magnet. It also depends upon the number of turns in the coil.

vi) The induced current exists so long as there is a relative motion between the coil and magnet.

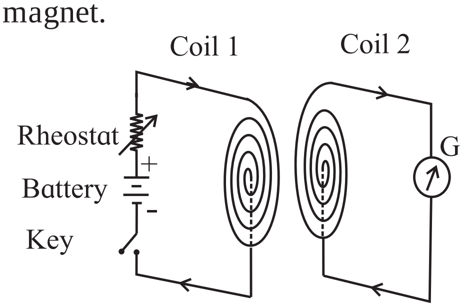


Fig. 12.2: Two coils with their planes facing each other.

vii) Instead of a magnet and a closed circuit, two coils with their planes facing each other (Fig. 12.2) also produce similar effects as mentioned above in experiments from (i) to (vi). One coil is connected in series with a battery, rheostat and key while the ends of the other coil are connected to a galvanometer (G). The coil which consists of a source of emf (a battery) is termed as primary coil while the other as secondary coil. With these two coils, following observations are made:

(i) When the circuit in the primary coil is closed or broken, a momentary deflection is produced in the galvanometer at the time of make or break. When the circuit is closed or broken the directions of deflection in the galvanometer are opposite to each other.

(ii) When there is a relative motion between the two coils (with their circuits closed), an induced current is produced in the secondary coil but it exists so long as there is a relative motion between the coils.

(iii) Whenever the current in the primary coil is changed (either increased or decreased) by sliding the rheostat-jockey, a deflection is produced in the galvanometer. This indicates the presence of induced current. The induced current exists so long as there is a change of current in the primary coil.

The above observations indicate that so long as there is a change of magnetic flux (produced either by means of a magnet or by a current carrying coil) inside a coil, an induced emf is produced. The direction of induced emf reverses if instead of increasing the flux, the flux is decreased or vice versa.

12.2 Faraday's Laws of Electromagnetic Induction:

On the basis of experimental evidences, Faraday enunciated following laws concerning electromagnetic induction.

First law: Whenever there is a change of magnetic flux in a closed circuit, an induced emf is produced in the circuit. Also, if a conductor cuts the lines of magnetic field, an e.mf. is induced between its ends.

This law is a qualitative law as it only indicates the characteristics of induced emf.

Second law: The magnitude of induced emf produced in the circuit is directly proportional to the rate of change of magnetic flux linked with the circuit. This law is known as quantitative law as it gives the magnitude of induced emf.

If ϕ is the magnetic flux linked with the coil at any instant t, then the induced emf.

$$e \propto \frac{\mathrm{d}\phi}{\mathrm{d}t} \quad \text{--- (12.1)}$$

$$e = K\frac{\mathrm{d}\phi}{\mathrm{d}t}, K \text{ is constant of proportionality.}$$

If e, ϕ, and t are measured in the same system of units, $K = 1$.

$$\therefore e = \frac{\mathrm{d}\phi}{\mathrm{d}t} \quad \text{--- (12.2)}$$

If we combine this expression with the Lenz's law (next article) , we get

$$e = -\frac{\mathrm{d}\phi}{\mathrm{d}t} \quad \text{--- (12.3)}$$

If ϕ' is the flux associated with single turn, then the total magnetic flux ϕ for a coil consisting of n turns, is

$$\phi = n\,\phi'$$

$$\therefore e = -n\frac{\mathrm{d}\phi'}{\mathrm{d}t} \quad \text{--- (12.4)}$$

This is also known as 'flux rule' according to which the emf is equal to the rate at which the magnetic flux through a conducting circuit is changing.

In SI units e is measured in volt and $\frac{\mathrm{d}\phi}{\mathrm{d}t}$ is measured in weber/s.

We have already learnt while studying the magnetic effect of current that the charges in motion (or current) can exert force/torque on a stationary magnet (compass meedle). Now we have observed in Faraday's experiments that a bar magnet in motion (or a time- varying magnetic field) can exert a force on the stationary charges inside the conductor and causes an induced emf across the ends of the conductor (open circuit)/or generates induced current in a closed circuit.

12.3 Lenz's Law:

H.F.E. Lenz (1804-1864) without knowledge of the work of Michael Faraday and Joseph Henry duplicated many of their discoveries independently almost at the same time.

For determining the direction of an induced current in a loop, Lenz devised a rule, which goes by his name as Lenz's Law. According to this rule, the direction of induced current in a circuit is such that the magnetic field produced by the induced current opposes the change in the magnetic flux that induces the current. The direction of induced emf is same as that of induced current. In short, the induced emf tends to set up a current the action of which opposes the change that causes it.

Applications of Lenz's law:

12.3.1 Motion of a Magnet Toward a Loop:

In order to get a feel for Lenz's law, let us consider a north pole of a magnet moving toward a conducting loop as shown in the Fig. 12.3.

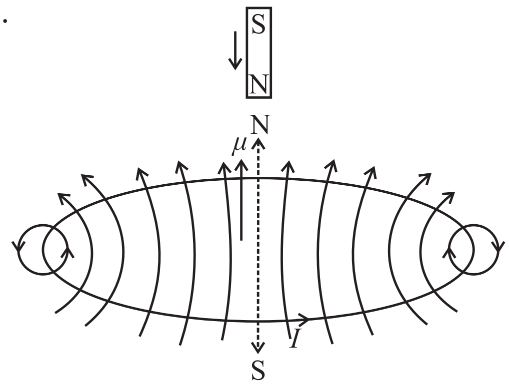


Fig. 12.3: Magnet's motion creates a magnetic dipole in the coil.

As the magnet is moved toward the loop, a current is induced in the loop. The induced current in the loop produces a magnetic dipole. The dipole is oriented in such a way that it opposes the motion of the magnet. Thus the loop's north pole must face the approaching north pole of the magnet so as to repel it. The curled right-hand (RH) rule for magnetic dipole or magnetic field will provide the direction of induced current in the loop. The induced current in the loop will be in counter clockwise direction as shown in Fig. 12.3.

If the magnet is pulled away from the loop, a current will again be induced in the loop in such a way that the loop will have a south pole facing the retreating north pole and will oppose the retreat by attracting it. The induced current in the loop will now flow in clockwise direction.

> **Jumping Ring Experiment:** A coil is wound around an iron core which is held vertically upright. A metallic ring is placed on top of the iron core. A current is then switched on to pass through the coil. This will make the ring jump several meters in air.
>
> **Explanation:** Before the current in the coil is turned on, the magnetic flux through the ring is zero. Afterwards, the flux appears in the coil in upward direction. This change in flux causes an induced emf and induced current as well in the ring. The direction of induced current in the ring will be opposite to the direction of current in the coil, as dictated by Lenz's law. As the opposite currents repel, the ring flies off in air.

12.3.2 Energy Conservation in Lenz's Law:

We have learnt that the cause of the induced current may be either (i) the motion of a conductor (wire) in a magnetic field or (ii) the change of magnetic flux through a stationary circuit.

In the first case, the direction of induced current in the moving conductor (wire) is such that the direction of the thrust exerted on the conductor (wire) by the magnetic field is opposite to the direction of its motion and thus opposes the motion of the conductor.

In the second case, the current sets up a magnetic field of its own which within the area bounded by the circuit is

(a) opposite to the original magnetic field if this field is increasing; but

(b) in the same direction as the original field, if the field is decreasing.

Thus it is the 'change in flux' through the circuit (not the flux itself), which is opposed by the induced current.

Lenz's law follows directly from the conservation of energy. If an induced current flows in a circuit in such a direction that it helps the cause that produces it, then we will soon find that the induced current and the magnetic flux penetrating the loop would lead to an infinite growth. The induced current once started flowing in the loop would keep increasing indefinitely producing joule heating at no extra cost and thus be self-sustaining (perpetual motion machine). This will violate the law of conservation of energy. We thus see that Lenz's law is a necessary consequence of the law of conservation of energy.

The opposing sense of the induced current is one manifestation of a general statement of Lenz's law: "Every effect of induction acts in opposition to the cause that produces it"

In order to have an induced current, we must have a closed circuit. If a conductor is not forming a closed circuit we mentally construct a circuit between the two ends of the conductor/ wire and use Lenz's law to determine the direction of induced current. Then the polarity of the ends of the open-circuited conductor can be found easily.

12.3.3 Lenz's Law and Faraday's Law:

Consider Faraday's law with special attention to the negative (-ve) sign.

$$e = -\frac{d\phi}{dt}.$$

Consider that area vector $\vec{A}$ of the loop perpendicular to the plane of the loop is fixed and oriented parallel ($\theta = 0$) to magnetic field $\vec{B}$. The magnetic field $\vec{B}$ increases with time.

Using the definition of flux, the Faraday' law can be written as

$$e = -\frac{d}{dt}(\vec{B}\cdot\vec{A}) = -|\vec{A}|\frac{d|\vec{B}|}{dt} \quad \text{--- (12.5)}$$

$\therefore$ RHS = -ve quantity as $|\vec{A}|$ is positive and $\frac{dB}{dt}$ is positive (+ve) as B is increasing with time.

The screw driver rule fixes the positive sense of circulation around the loop as the clockwise direction.

As the sense of the induced current in the loop is counter clockwise (negative), the sense of induced emf also is negative (-ve). That is, the LHS of Eq. (12.5) is indeed a negative (-ve) quantity in order to be equal to the RHS.

Thus the negative (-ve) sign in the equation $e = -\frac{d\phi}{dt}$ incorporates Lenz's law into Faraday's law.

12.4 Flux of the Field:

The concept of flux of the magnetic field is vital to our understanding of Faraday's law.

As shown in Fig. 12.4 (a), consider a small element of area $\vec{da}$. A direction is assigned to this element of area such that if the curve bounding the area is traversed in the direction of the arrow then the normal comes out of the plane of paper towards the reader. In other words it is the direction in which right handed screw will move if rotated in the sense of the arrow on the curve.

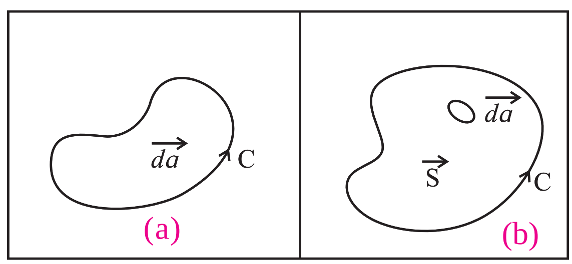


Fig. 12.4: (a) Small element of area $\vec{da}$ bounded by a curve considered in anticlockwise direction. (Right-handed screw Rule), (b) Finite surface area $\vec{S}$.

Suppose the element of area $\vec{da}$ is situated in a magnetic field $\vec{B}$. Then the scalar quantity

$$d\phi = \vec{B}.\vec{da} = |\vec{B}|.|\vec{da}|\cos\theta \quad \text{--- (12.6)}$$

is called the flux of $\vec{B}$ through the area $\vec{da}$ where θ is the angle between the direction of magnetic field $\vec{B}$ and the direction assigned to the area $\vec{da}$.

This can be generalised to define the flux over a finite area $\vec{S}$. It should be remembered that the magnetic field $\vec{B}$ will not be the same at different points within the finite area. Therefore the area is divided into small sections of area $\overrightarrow{da}$ so as to calculate the flux over each section and then to integrate over the entire area (Fig. 12.3 (b))

Thus, the flux passing through S is

$$\phi = \int_S \vec{B}\cdot\overrightarrow{da} \quad \text{--- (12.7)}$$

We can not take $\vec{B}$ out of the integral in Eq. (12.6) unless $\vec{B}$ is the same everywhere in $\vec{S}$.

If the magnetic field at every point changes with time as well, then the flux ϕ will also change with time.

$$\phi = \phi(t) = \int_S \vec{B}(t)\cdot\overrightarrow{da} \quad \text{--- (12.8)}$$

Faraday's discovery was that the rate of change of flux $\left(\frac{d\phi}{dt}\right)$ is related to the work done to take a unit positive charge around the contour C [Fig. 12.4 (b)] in the 'reverse' direction. This work done is just the emf.

Accordingly, Faraday's law states that the induced emf can be written as

$$e = -\frac{d\phi}{dt} = -\frac{d}{dt}\int_S \vec{B}(t)\cdot\overrightarrow{da} \quad \text{--- (12.9)}$$

In S.I. units the emf, *e* will be in volt, the flux ϕ in weber and time *t* in second.

From Eq. (12.9) we can see that even if $\vec{B}$ does not change with time, flux may still vary if the area S changes with time.

12.5 Motional Electromotive Force:

a) Translational motion of a conductor:

As shown in Fig. 12.5, a C shaped frame of wires ABCD of area (l x) is situated in a constant magnetic field ($\vec{B}$) BC is conducting wire that slides on the frame parallel to AD. As the wire BC of length l is moved out with velocity $\vec{v}$ to increase x the area of the loop ABCD increases. Thus the flux of $\vec{B}$ through the loop increases with time. According to the 'Flux Rule' the induced emf will be equal to the rate at which the magnetic flux through a conducting circuit is changing as stated in Eq. (12.9). The induced emf will cause a current in the loop. It is assumed that there is enough resistance in the wire so that the induced currents are very small producing negligible magnetic field.

As the flux ϕ through the frame ABCD is Blx, magnitude of the induced emf can be written as

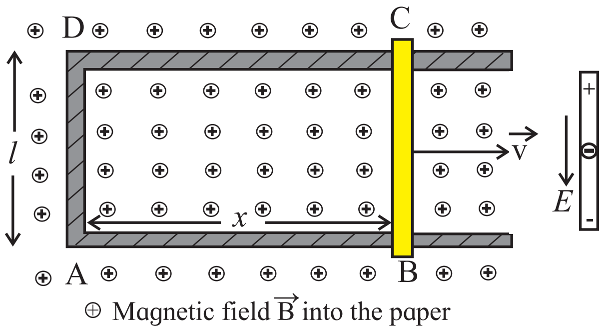


Fig. 12.5: A frame of wire ABCD in magnetic field $\vec{B}$. Wire BC is moving with velocity $\vec{v}$ along x- axis.

$$|e| = \frac{d\phi}{dt} = \frac{d}{dt}(Blx) = Bl\frac{dx}{dt} = Blv \quad \text{,--- (12.10)}$$

where v is the velocity of wire BC increasing the length x of wires AB and CD.

Now we can understand the above result from the magnetic forces on the charges in the moving wire BC.

A charge q which is carried along by the moving wire BC, experiences Lorentz force $\vec{F} = q(\vec{v}\times\vec{B})$; which is perpendicular to both $\vec{v}$ and $\vec{B}$ and hence is parallel to wire BC. The force $\vec{F}$ is constant along the length l of the wire BC (as v and B are constant) and zero elsewhere ($\because$ v = 0 for stationary part CDAB of wire frame). When the charge q moves a distance l along the wire, the work done by the Lorentz force is $W = F\cdot l = qvB\sin\theta . l$, where θ is the angle between $\vec{B}$ and $\vec{v}$. The emf generated is work/ charge i.e.,

$$e = \frac{W}{q} = vB\sin\theta\cdot l \quad \text{--- (12.11)}$$

For maximum induced emf, $\sin\theta = 1$

$$e_{max} = Blv \quad \text{--- (12.12)}$$

which is the same result as obtained in Eq. (12.10) derived from the rate of change of flux.

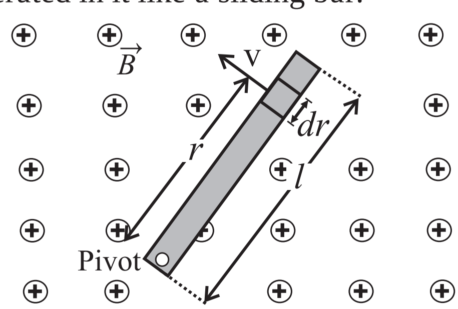
In general, it can be proved that for any circuit whose parts move in a fixed magnetic field, the induced emf is the time derivative of flux (ϕ) regardless of the shape of the circuit.

The flux rule is also applicable in case of a wire loop that is kept stationary and the magnetic field is changed. The Lorentz force on the electrical charges is given by $\vec{F} = q\,(\vec{E} + \vec{v} \times \vec{B})$. There are no new special forces due to changing magnetic fields. Any force on charges at rest in a stationary wire comes from the $\vec{E}$-term. Faraday's observations led to the discovery that electric and magnetic fields are related by a new law: *In a region where magnetic field is changing with time, electric fields are generated*. It is this electric field which drives the electrons around the conductor circuit and as such is responsible for the induced emf in a stationary circuit whenever there is a changing magnetic flux.

The flux rule holds good so long as the change in the magnetic flux is due to the changes in magnetic field or due to the motion of the circuit or both.

b) Motional emf in a rotating bar:

A rotating bar is different in nature from the sliding bar. As shown in Fig. 12.6, consider a small segment *dr* of the bar at a distance *r* from the pivot. It is a short length *dr* of the conductor which is moving with velocity $\vec{v}$ in magnetic field $\vec{B}$ and has an induced emf generated in it like a sliding bar.



⊕ Magnetic field $\vec{B}$ into the paper

Fig. 12.6: A conducting bar rotating around a pivot at one end in a uniform magnetic field that is perpendicular to the plane of rotation. A rotational emf is induced between the ends of the bar.

By imagining all such segments as a source of emf, we can find that all these segments are in series and, therefore, the emfs of individual segments will be added.

Now we know that the induced emf *de* in the small segment *dr* of the rotating conductor.

$de = B\,v\,dr$

Total induced emf in rotating rod

$$e = \int de = \int Bv dr$$

$$e = \int B\omega r dr = B\omega \int_0^l r dr$$

$$= B\omega \frac{l^2}{2}$$

$$e = \frac{1}{2} B\omega l^2 \qquad \text{--- (12.13)}$$

Compare the above result with the induced emf in sliding bar, $e = Blv$.

12.6 Induced emf in a Stationary Coil in a Changing Magnetic Field:

As shown in Fig. 12.7 (a) in a magnet-coil system, a permanent bar magnet is mounted on an arc of a semicircle of radius 50 cm. The arc is a part of a rigid frame of aluminium and is suspended at the centre of arc so that whole system can oscillate freely in its plane. A coil of about 10,000 turns of copper wire loop the arc so that the bar magnet can pass through the coil freely.

When the magnet moves through the coil, the magnetic flux through the coil changes.

In order to measure the induced emf, a capacitor (C) and diode (D) are connected across the coil (Fig. 12.7 (b)) The induced emf produced in the coil is used for charging a capacitor through a diode. Then the voltage developed across the capacitor is measured. The capacitor may not get charged upto the peak value in a single swing as the time-constant (RC) may be larger than the time during which the emf in the coil is generated. This may take about a few oscillations to charge the capacitor to the peak value and is indicated by the ammeter (mA) which will tell us when the charging current ceases to flow.

As the magnet, kept in the middle of the arc (Fig. 12.7 (a)), starts far away from the coil moves through it and recedes, the magnetic field /flux through the coil changes from a small value, increases to its maximum and becomes small again thus inducing an emf. Actually, there is substantial magnetic field at the coil only when it is very near the magnet. The speed of the magnet is largest when it approaches the coil (placed at the mean position of the oscillation). Thus the magnetic field changes quite slowly with time when the magnet is far away and changes rapidly when it approaches the coil. The variation of magnetic field $\vec{B}$ (at the coil in mean position) with time is shown in Fig. 12.7 (c).

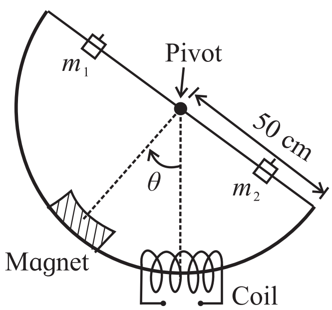


Fig. 12.7 (a): Magnet-coil system.

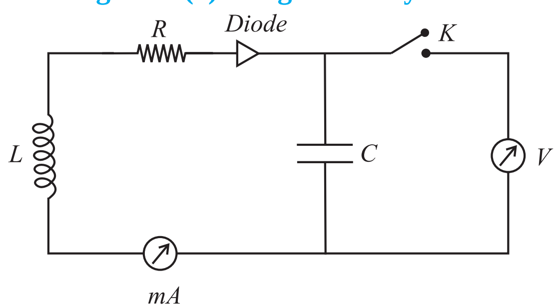


Fig. 12.7 (b): Measurement of induced emf.

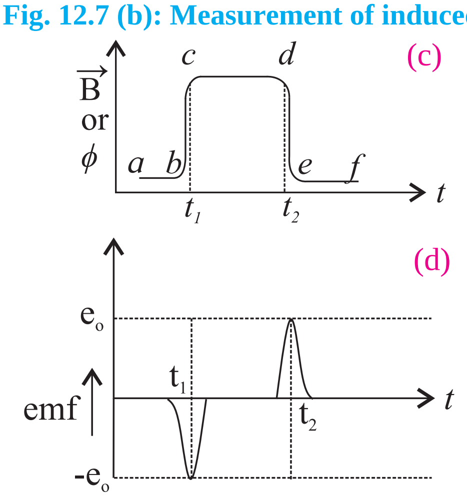


Fig. 12.7: (c) Variation of $\vec{B}$ with time *t*, (d) variation of *e* with time *t*.

The centre of the hump 'bcde' refers to the time when the magnet is inside the coil. The flat portion (cd) at the top corresponds to the finite length of the magnet. The magnetic flux (ϕ) is related to magnetic field (B) through a constant (effective area = No. of turns × area of coil).

Now, the induced emf is proportional to $d\phi/dt$, that is to the slope of the curve in Fig. 12.7 (c). As the slope of the curve is largest at times t_1 and t_2, the magnitude of induced emf will be largest at these times. But Lenz's law gives minus sign (-) in Eq. 12.3 $\left(e = -\dfrac{d\phi}{dt}\right)$, which means that emf ($e$) is 'negative' when ϕ is increasing at t_1 and 'positive' when ϕ is decreasing at time t_2. This is shown in Fig. 12.7 (d) relating induced emf (e) with time (t).

Remember the sequence of two pulses; one 'negative' and one 'positive' occurs during just half a cycle of motion of the magnet. On the return swing of the magnet, they will be repeated (which one will be repeated first, the 'negative' or 'positive' pulse?).

Now we consider the effect of these pulses on the charging circuit (Fig. 12.7 (b)) The diode will conduct only during the 'positive' pulse. At the first half swing, the capacitor will charge up to a potential, say e_1. During the next half swing, the diode will be cut off until 'positive' pulse is produced and then the capacitor will charge upto a slightly higher potential, say e_2. This will continue for a few oscillations till the capacitor charges upto its peak value e_o by the voltage/ emf pulse. At this stage ammeter will show no kick (further increase) in the current of the circuit.

In order to have an estimate of e_o, the equation for induced emf can be written as

$$|e| = \left|\frac{d\phi}{dt}\right| = \left|\frac{d\phi}{d\theta}\right| \cdot \left|\frac{d\theta}{dt}\right| \qquad \text{--- (12.14)}$$

The first term depends on the geometry of the magnet and the coil. At $\theta = 0$, the mean position, we have maximum ϕ. But we are interested in $\left(\frac{d\phi}{d\theta}\right)$, which is actually zero at $\theta = 0$. The second term $\frac{d\theta}{dt}$ can be deduced from the oscillation equation.

$\theta = \theta_0 \sin 2\pi\nu t$, θ_0 being the amplitude of oscillating magnet.

$$\because \text{frequency}\,(\nu) = \frac{1}{\text{time period}\,(T)}$$

$$\theta = \theta_0 \sin \frac{2\pi}{T} t$$

$$\therefore \frac{d\theta}{dt} = \theta_0 \cos \frac{2\pi}{T} t \cdot \left(\frac{2\pi}{T}\right)$$

$$\frac{d\theta}{dt} = \frac{2\pi\theta_0}{T} \cos \frac{2\pi t}{T} \qquad \text{--- (12.15)}$$

The peak voltage (emf) e_0 in the induced emf pulse corresponds to $\left(\frac{d\phi}{dt}\right)_{max}$.

We can see from Fig. 12.7 (c) that $\left(\frac{d\phi}{d\theta}\right)_{max}$ occurs at positions near the mean position. In Eq. 12.15, the cosine term does not differ much from unity for very small angles (close to zero).

Hence we conclude that

$$|e_0| = \left(\frac{d\phi}{dt}\right)_{max} \approx \left(\frac{d\phi}{d\theta}\right)_{max} \cdot \left(\frac{2\pi\theta_0}{T}\right) \text{--- (12.16)}$$

For given magnet-coil system, the peak induced emf e_0 is directly proportional to angular amplitude (θ_0) and inversely proportional to time period (T).

Example 12.1: A coil consists of 400 turns of wire. Each turn is a square of side $d = 20$ cm. A uniform magnetic field directed perpendicular to the plane of the coil is turned on. If the field changes linearly from 0 to 0.50 T in 0.8 s, what is the magnitude of induced emf in the coil while the field is changing?

Solution: The magnitude of induced emf in the coil is written as

$|e| = |d(N\phi)/dt| = N\,(d\phi/dt)$

$\because \phi = B \cdot A$

$\therefore |e| = N \cdot d(BA)/dt$

$= N \cdot A \cdot (dB/dt)$ (as A is constant and B is changing with time)

$= N \cdot A \cdot (\Delta B / \Delta t) = N \cdot A \cdot (B_{final} - B_{initial} / \Delta t)$

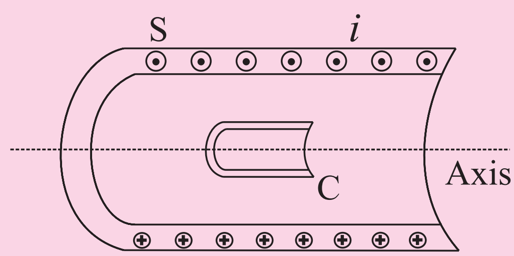
Inserting the given values of N=400, A = (20 cm)^2 = (0.2 m)^2, B_{final} = 0.5 T, $B_{initial}$ = 0, Δt = 0.8 s

we find the induced emf

$|e|$ = 400·(0.20 m)^2 (0.5 T)/0.8 s

= 10 Volt

Example 12.2 : A long solenoid S, as shown in the figure has 200 turns/cm and carries a current i of 1.4 A. The diameter D of the solenoid is 3 cm. A coil C, having 100 turns and diameter d of 2 cm is kept coaxial to the solenoid. The current in the solenoid is decreased steadily to zero in 20 ms. Calculate the magnitude of emf induced in the coil C when the current in the solenoid is changing.



⊙ Magnetic field $\vec{B}$ out of paper

⊕ Magnetic field $\vec{B}$ into the paper

Solution: Part of magnetic flux (per turn) of the solenoid S that links with the coil C is

$$\phi_c = \mu_0 n_s i \frac{\pi d_c^2}{4}$$

This flux reduces to zero in dt =20ms. Thus, the emf induced in coil C of N_c turns is

$$e_s = -N_c \frac{d\phi_c}{dt} = \frac{-(0-\phi_c)}{dt} = \frac{\mu_0 N_c n_s i \pi d_c^2}{4dt}$$

$$= \frac{4\pi\times10^{-7}\times100\times2\times10^4\times1.4\times3.14\times10^{-4}}{4\times20\times10^{-3}}$$

= 55.24 mV

12.7 Generators:

In Chapter 10 you have learnt the principle of electric motors. The basic construction of an electric generator is the same as that of a motor. In this case the armature is turned by some external agency/torque as shown in Fig.12.8 (a). As the conductor wires cut across the magnetic lines of force, an induced emf ($e = Blv$) is produced across the terminals of the commutator. The induced e.m.f is found to be proportional to the speed of rotation (ω) of the armature.

Let us focus our attention on one conductor of the armature as shown in Fig. 12.8 (b). In position (i), the conductor is moving upward across the lines of force inducing maximum emf. When the armature reaches in position (ii) the conductor is moving parallel to the field and there is no induced emf (e = 0). At position (iii), the same conductor moves down across the lines of force and the induced emf/ current is directed opposite to that in case of (i). The graph, plotted between the current flowing in the lamp as a function of the time (t) shows a sinusoidally varying current as is shown in (iv) of Fig. 12.8 (b).

When a coil is rotating with a constant angular velocity ω, the angle between magnetic field $\vec{B}$ and the area vector $\vec{A}$ of the coil at any instant t is $\theta = \omega t$ (assuming $\theta = 0$ at $t = 0$). As the effective area of the coil is changing due to rotation in the magnetic field B, the flux ϕ_B at any time can be written as

$$\phi_B = B{\cdot}A \cos\theta = B{\cdot}A \cos \omega t.$$

From Faraday's law, the induced emf e, generated by a rotating coil of N turns

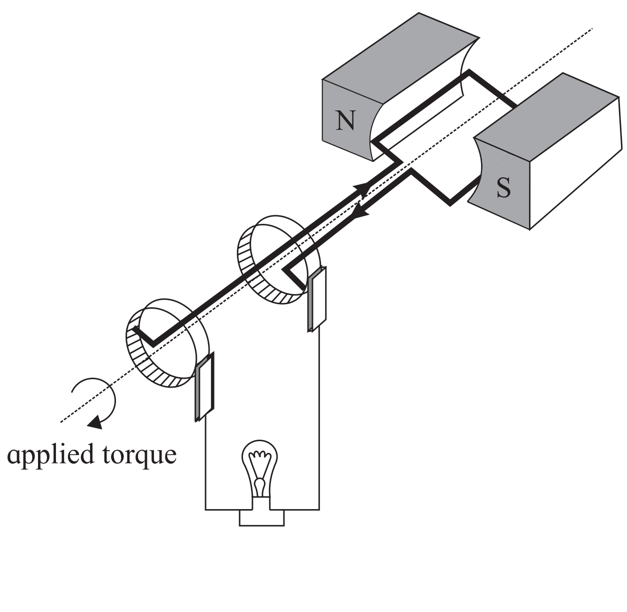


Fig. 12.8 (a): Schematic of a Generator.

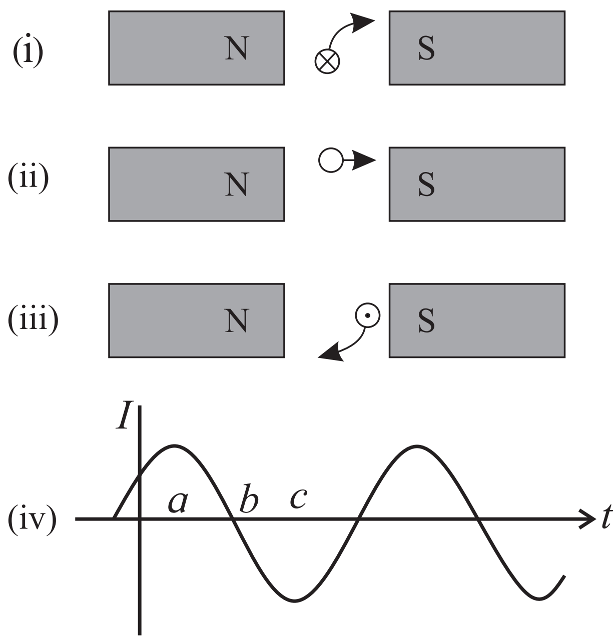


Fig. 12.8 (b): Wave form generation.

$$e = -\frac{Nd\phi_B}{dt}$$

$$= -N \cdot \frac{d}{dt}(BA\cos\omega t)$$

$$= NBA\omega \ \sin\omega t$$

For $\sin\omega t = \pm 1$

$$e = \pm NBA\omega = \pm e_0$$

$$\therefore e = e_0 \sin\omega t$$

$$e = e_0 \sin 2\pi ft, \qquad \text{--- (12.17)}$$

where f is the frequency of revolution of the coil.

Since the value of sin ωt varies between +1 and -1, the polarity of the emf changes with time. The emf has its extremum value at θ = 90° and 270° as the change in flux is greatest at these points. As the direction of induced current changes periodically it is called as alternating current (AC) (Fig. 12.8 (c)). The frequency of AC is equal to the number of times per second, the current changes from positive (+ve) to negative (-ve) and back again. The domestic electrical current varies at a frequency of 50 cycles/second.

For the purpose of charging a storage

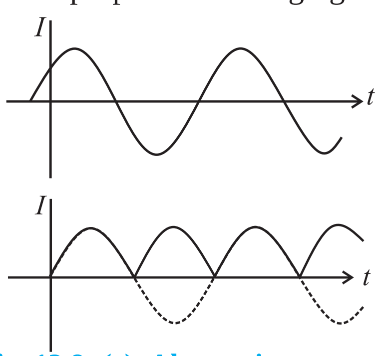
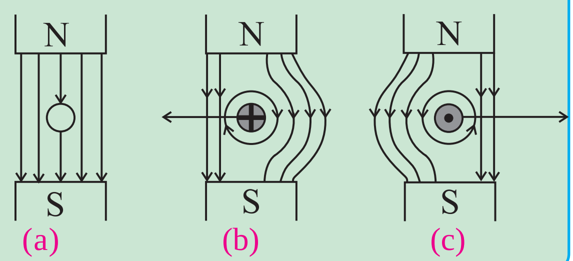


Fig. 12.8: (c) Alternating current,
(d) Pulsating direct current.

battery it is necessary to generate a steady or direct current (DC). The reversing action of commutator can be used to generate pulsating DC as depicted in Fig. 12.8 (d). The commutator acts like a rapid switch which reverses the connections to the armature at just the right times to match with the reversals in current. Modern AC motors are more compact and rugged than the DC motors.

Do you know?

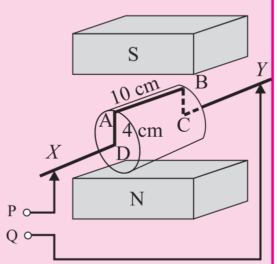
If a wire without any current is kept in a magnetic field, then it experiences no force as shown in figure (a). But when the wire is carrying a current into the plane of the paper in the magnetic field, a force will be exerted on the wire towards the left as shown in the figure (b). The field will be strengthened on the right side of the wire where the lines of force are in the same direction as that of the magnetic field and weakened on the left side where the field lines are in opposite direction to that of the applied magnetic field. For a wire carrying a current out of the plane of the paper, the force will act to the right as shown in figure (c).



12.8 Back emf and back torque:

We know that emf can be generated in a circuit in different ways. In a battery it is the chemical force, which gives rise to emf. In piezoelectric crystals mechanical pressure generates the emf. In a thermocouple it is the temperature gradient which is responsible for producing emf in a circuit containing the junctions of two metallic wires. In a photo electric cell, the incident light above a certain frequency is responsible for producing the emf. In a Van de Graaff Generator the electrons are literally loaded into a conveyor belt and swept along to create a potential difference. A generator utilises the movement of wire through a magnetic field to produce motional emf/current through a circuit. We have seen that the physical construction of a DC generator and motor is practically the same. If a DC generator is connected to a battery, it will run as a motor. If a motor is turned by any external means, it will behave as a generator. So whenever a motor is running, its generator action can not be turned off. By Lenz's law the induced emf will tend to oppose the change which causes it. In the present case, the 'cause' is the current through the armature. Therefore, the induced emf will tend to reduce the armature current. The induced emf which is unavoidable due to generator action in a motor is called back emf. Initially, when a motor is just starting up, its armature is not turning and hence it is not producing any back emf. As the motor starts speeding up the back emf increases and armature current decreases. This explains the reason as to why the current through a motor is larger in the beginning than when the motor is running at full speed.

Example 12.3: A rotating armature of a simple generator consists of rectangular section DABC of a conducting wire as shown in the figure, to which connections are made through sliding contacts. The armature is rotated at 1500 rpm in the magnetic field ($\vec{B}$) of 0.5 N/ A.m. Determine the induced emf between the terminals P and Q of the generator at the instant shown in the adjoining figure.



Solution: The wire AB (l = 10 cm) is moving to the right with the tangential velocity v.

$$|\vec{v}| = |\vec{\omega} \times \vec{r}|$$

v = ωr where ω is angular velocity and r is the radius.

$$v = \frac{2\pi}{T}.r \quad \left[\because \omega = \frac{2\pi}{T}\right]$$

$$= 2\pi v r$$

$$= 2\pi \cdot \left(\frac{1500}{60\,\text{s}}\right) \cdot \left(\frac{4}{100}\right) m \quad \left[\begin{array}{l}\because r = AD \\ = (4/100)\text{m}\end{array}\right]$$

$$= 2\pi \ \ \text{m/s}$$

$$= 6.284 \text{ m/s}$$

The magnetic field is directed vertically upward from North to South pole. As the wire AB is cutting the magnetic lines of force perpendicularly, the induced emf is, therefore, maximum.

$\because e = Blv \sin\theta$ with $\theta = 90°$,

Then, $e = e_{max} = Blv$

$= (0.5 \text{ N/A.m}) (10/100)\text{m}. (6.28\text{m/s})$

$$= \frac{0.5}{10} \times 6.28$$

$$= 0.314 \text{ V}$$

or $e_{max} \approx 314.2\ mV$.

The emf induced in the wires BC and DA is zero because the magnetic Lorentz force on free electrons in these wire $\left[\vec{F} = q(\vec{v} \times \vec{B})\right]$ has no component parallel to the wires. Also there is no e.m.f. in the lead in wires, which are stationary and are not in motion ($\vec{v} = 0$). Therefore the total emf between the terminals P and Q is due the movement of segment AB. i.e., e = 314 mV. The direction of induced emf is given by Lenz's law.

Example 12.4: A conducting loop of area 1 m^2 is placed normal to uniform magnetic field 3Wb/m^2. If the magnetic field is uniformly reduced to 1 Wb/m^2 in a time of 0.5 s, calculate the induced emf produced in the loop.

Solution: Given,

Area of the loop, A = 1 m^2

$(B)_{intial}$ = 3Wb/m^2

$(B)_{final}$ = 1Wb/m^2

duration of time, Δt = 0.5 s

$\therefore$ Induced emf,

$$|e| = \left|\frac{d\phi}{dt}\right| = \left|\frac{\phi_{final} - \phi_{initial}}{\text{Time interval}}\right|$$

$$= \frac{(B_{final} - B_{initial})A}{\Delta t}$$

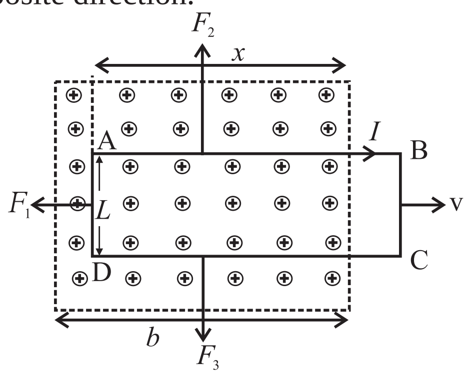
$$= \left|\left[\frac{(1-3)}{0.5} \cdot 1\right]\right| \text{volt}$$

$$= \left|-\frac{2}{0.5} \cdot 1\right| \text{volt}$$

$$|e| = 4 \text{ volt}$$

12.9 Induction and Energy Transfer:

Consider a loop ABCD taken out with constant velocity $\vec{v}$ through a uniform magnetic field $\vec{B}$ as shown in Fig. 12.9 (a). A current i is induced in the loop in clockwise direction and the loop segments, being still in magnetic field, experience forces, F_1, F_2 and F_3. The dashed lines show the limits of magnetic field. To maintain the velocity $\vec{v}$ constant, it is required to apply an external force $\vec{F}$ on the loop so as to overcome the magnetic force of equal magnitude but acting in opposite direction.



⊕ Magnetic field $\vec{B}$ into plane of the paper

Fig. 12.9 (a): A loop is moving out of magnetic field with velocity v.

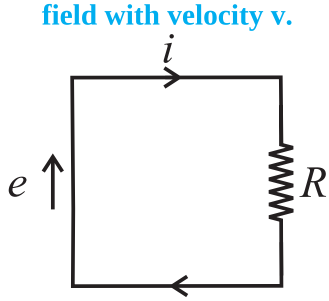


Fig. 12.9 (b) : Induced emf e, induced current i and collective resistance R of the loop.

$\therefore$ The rate of doing work on the loop is

$$P = \frac{\text{Work }(W)}{\text{time }(t)} = \frac{\text{Force }(F)\times\text{displacement }(d)}{\text{time }(t)}$$

$$P = \text{Force }(F)\times\text{velocity }(\text{v})$$

$$= \vec{F}\cdot\vec{\text{v}} \qquad \text{--- (12.18)}$$

We would like to find the expression for P in terms of B and the characteristics of the loop i.e., resistance (R), width (L) and Area (A).

As the loop is moved to the right, the area lying within the magnetic field decreases, thus causing a decrease in the magnetic flux linked with the moving loop. The decreasing magnetic flux induces current in the loop as dictated by Lenz's law. The induced current in the loop gives rise to a force that opposes the pulling of the loop out of the magnetic field.

We know that magnitude of magnetic flux through the loop is

$$\phi_B = B.A = B.L.x \qquad \text{--- (12.19)}$$

As x decreases, the flux decreases. According to Faraday's law, the magnitude of induced emf,

$$|e| = \left|\frac{\mathrm{d}\phi}{\mathrm{d}t}\right| = \frac{\mathrm{d}}{\mathrm{d}t}(BLx)$$

$$= BL\cdot\frac{\mathrm{d}x}{\mathrm{d}t} = BL\text{v} \qquad \text{--- (12.20)}$$

The induced emf e is represented on the left and the collective resistance R of the loop on the right in the Fig. 12.9 (b). The direction of induced current i is obtained by Right-Hand (RH) Rule.

The magnitude of induced current i can be written using Eq. (12.20) as

$$i = \frac{|e|}{R} = \frac{BL\text{v}}{R} \qquad \text{--- (12.21)}$$

The three segments of the current carrying loop experience the deflecting forces $\vec{F}_1$, $\vec{F}_2$ and $\vec{F}_3$ in the magnetic field $\vec{B}$ in accordance with Eq. ($\vec{F} = i\,\vec{L}\times\vec{B}$). From the symmetry, the forces $\vec{F}_2$ and $\vec{F}_3$ being equal and opposite, cancel each other. The remaining force $\vec{F}_1$ is directed opposite to the external force $\vec{F}$ on the loop. So $\vec{F} = -\,\vec{F}_1$.

The magnitude of $|\vec{F}_1|$ can be written as

$$|\vec{F}_1| = i\,LB\sin 90 = i\,LB = |\vec{F}| \qquad \text{--- (12.22)}$$

From Eq. (12.21) and Eq. (12.22)

$$|\vec{F}| = |\vec{F}_1| = iLB$$

$$= \frac{BL\text{v}}{R}\cdot LB = \frac{B^2L^2\text{v}}{R} \qquad \text{--- (12.23)}$$

From Eq. (12.18) and (12.23), the rate of doing mechanical work, that is power:

$$P = \vec{F}.\vec{\text{v}} = \frac{B^2L^2\text{v}}{R}\cdot\text{v} = \frac{B^2L^2\text{v}^2}{R} \qquad \text{--- (12.24)}$$

If current i is flowing in the closed circuit with collective resistance R, the rate of production of heat energy in the loop as we pull it along at constant speed v, can be written as

Rate of production of heat energy =

$$P = i^2 R \quad \text{--- (12.25)}$$

From Eq. (12.21) and Eq. (12.25)

$$P = \left(\frac{BL\text{v}}{R}\right)^2 \cdot R$$

$$P = \frac{B^2 L^2 \text{v}^2}{R} \quad \text{--- (12.26)}$$

Comparing Eq. (12.24) and Eq. (12.26), we find that the rate of doing mechanical work is exactly same as the rate of production of heat energy in the circuit/loop.

Thus the work done in pulling the loop through the magnetic field appears as heat energy in the loop.

12.10 Eddy Currents:

Suppose the conducting loop of Fig. 12.9 (a) is replaced by a solid conducting plate, the relative motion between conductor and magnetic field induces a current in the conductor plate (Fig. 12.10 (a)). In this case again, we encounter an opposing force so we must do work while moving the conductor with uniform velocity v. The conduction electrons making up the induced current do not follow one path as they do with the loop, but swirl about within the plate as if they were caught in an eddy of water. Such a current is called an eddy current. Eddy current can be represented by a single path as shown in Fig. 12.10 (a).

The induced current in the conductor plate is responsible for transfer of the mechanical energy into heat energy. The dissipation of energy as heat energy is more apparent in the arrangement shown in Fig. 12.10 (a), where a conducting plate, free to rotate about a pivot, is allowed to swing down like a pendulum through a magnetic field. In each swing, when the plate enters and leaves the field, a portion of its mechanical energy is transformed to heat energy. After several such swings there is no mechanical energy left with the pendulum and the converted heat energy is dissipated in the solid plate making it warm. Eddy current can be reduced by discontinuity in the structure of conductor plate as depicted in Fig. 12.10 (c).

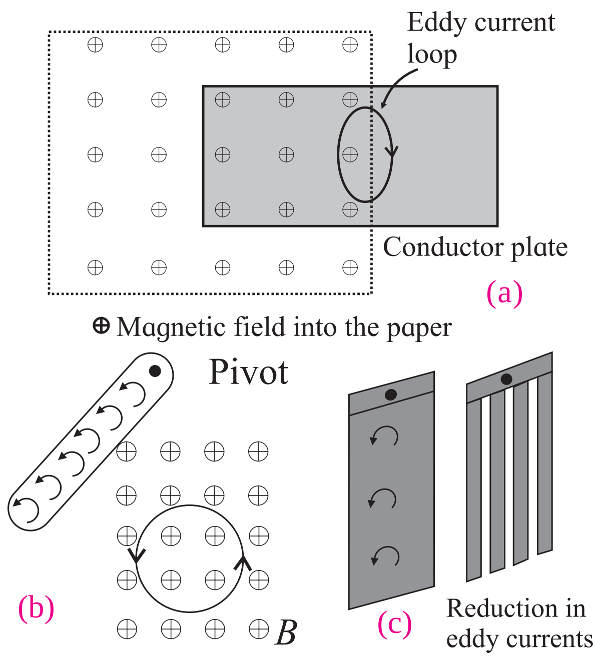


Fig. 12.10: (a) Eddy currents are induced in solid conductor plate, (b) Conducting plate swings like a pendulum, (c) Reduction in eddy currents due to discontinuous structure of a plate.

12.11 Self-Inductance:

Consider a circuit (coil) in which the current is changing. The changing current will vary the magnitude of magnetic flux linked with the coil (circuit) itself and consequently an emf will be induced in the circuit.

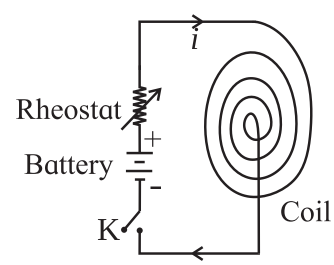


Fig. 12.11: Changing current in a coil.

The production of induced emf, in the circuit (coil) itself, on account of a change in the current in it, is termed as the phenomenon of self-inductance.

Let at any instant, the value of magnetic flux linked with the circuit itself be ϕ

corresponding to current i in it (Fig. 12.11). It is obvious that ϕ will be proportional to current i.

i.e., $\phi \propto i$

or $\phi = Li$ or $L = \phi / i$, --- (12. 27)

where L is a constant of proportionality and is termed as the self-inductance (or coefficient of self induction) of the coil.

For a closely wound coil of N turns, the same magnetic flux will be linked with all the turns. When the flux through the coil changes each turn of the coil contributes towards the induced emf. Therefore a term flux linkage is used for a closely wound coil. The flux linkage for a coil with N turns corresponding to current i will be written as

$$N\phi_B \propto i$$

$$N\phi_B = Li$$

$$L = N\phi_B / i \quad \text{--- (12.28)}$$

The inductance (L) depends only on the geometry and material properties of the core of the coil.

Unit of Inductance:

According to Faraday's law, induced emf e is given by

$$e = -\frac{d\phi}{dt}$$

Using Eq. (12.27)

$$e = -\frac{d}{dt}(Li) = -L\frac{di}{dt} \quad \text{--- (12.29)}$$

Unit of $L = \frac{|e|}{|di/dt|} = \left[\frac{\text{volt}}{\text{A/s}}\right] = \text{Henry}$

Definition of L:

Self inductance L may be defined in the following ways:

(i) From Eq. (12.27), $\phi = Li$ or $L = \frac{\phi}{i}$

Hence, the self-inductance of a circuit is the ratio of magnetic flux (produced due to current in the circuit) linked with the circuit to the current flowing in it. The magnetic flux produced per unit current in the circuit is defined as its self inductance.

(ii) Using Eq. (12.29), $L = -\left(\frac{e}{di/dt}\right)$

Hence, self-inductance of a circuit is the ratio of induced emf (caused by changing current in the circuit) produced around the circuit to the rate of change of current in it.

In order words, the induced emf produced around the circuit per unit rate of change of current in it, is defined as the self-inductance of the circuit.

(iii) When a current increases in the circuit, an induced emf acts opposite to it. Consequently, the work will have to be done in order to establish the magnetic flux associated with a steady current i_o in the circuit.

Work done in time dt is d$W = e.i$ dt

$$= \left(-L\frac{di}{dt}\right)\cdot(i)\cdot dt\left[\because e = -L\frac{di}{dt}\right]$$

$$= -Li\frac{di}{dt}\cdot dt$$

$$= -Li \cdot di$$

$\therefore$ Total work $W = \int dw = \int_o^{i_o} -Lidi$

or, $$W = -L\cdot\frac{i_0^2}{2} \quad \text{--- (12.30)}$$

$$= \frac{1}{2}Li_0^2 \quad \text{(magnitude)}$$

Now if $i_0 = 1$,

Then $W = L\cdot\frac{1}{2}$

or $L = +2W$ (numerically) --- (12.31)

Hence self-inductance of a circuit is numerically equal to twice the work done in establishing the magnetic flux associated with unit current in the circuit.

This work done W, will represent the energy of the circuit.

$\therefore$ Energy of the circuit $= \frac{1}{2}Li_0^2$ --- (12.32)

We know that the mechanical energy is expressed in terms of kinetic energy as

$$\text{KE} = \frac{1}{2}mv^2 \quad \text{--- (12.33)}$$

Comparing the Eq. (12.33) and Eq. (12.34), we find that self inductance (L) of an electrical circuit plays the same role (electrical inertia) as played by mass (inertia) in mechanical motion.

Inductance of a solenoid: If a current i is established in the windings (turns) of a long solenoid, the current produces a magnetic flux ϕ_B through the central region. The inductance of the solenoid is given by $L = N\phi_B/i$, where N is the number of turns. $N\phi_B$ is called as magnetic flux linkage. For a length l near the middle of the solenoid the flux linkage is $N\phi_B = (nl)(\vec{B}.\vec{A}) = nlBA$, (for $\theta = 0°$), where n is the number of turns per unit length, B is the magnetic field inside and A is the cross sectional area of the solenoid.

We know that the magnetic field inside the solenoid is given by Eq. (10.65) as

$$B = \mu_0 ni$$

Hence

$$L = \frac{N\phi_B}{i} = \frac{(nl)BA}{i} = \frac{nl(\mu_0 ni)A}{i}$$

$$= \mu_0 n^2 lA$$

where, Al is the interior volume of solenoid.

Therefore inductance per unit length near the middle of a long solenoid is

$$\frac{L}{l} = \mu_0 n^2 A \quad = \mu_0 n^2 \left(\frac{\pi d^2}{4}\right),$$ d being the diameter of solenoid. --- (12.34)

This implies that inductance of a solenoid $L \propto n^2$, $L \propto d^2$. As n is a number per unit length, inductance can be written as a product of permeability constant μ_0 and a quantity with dimension of length. This implies that μ_0 can be expressed in henry/ meter (H/m).

Example 12.5: Derive an expression for the self-inductance of a toroid of circular cross-section of radius r and major radius R. Calculate the self inductance (L) of toroid for major radius (R) = 15 cm, cross-section of toroid having radius (r) =2.0 cm and the number of turns (n) =1200.

Solution: The magnetic field inside a toroid,

$$B = \frac{\mu_0 Ni}{2\pi r_0}$$, where N is the number of turns and r_0 is the distance from the toroidal axis. If $r << R$, we can use $r_0 \cong$ R. Hence,

$$B = \frac{\mu_0 Ni}{2uR}$$

The magnetic flux (ϕ) passing through cavity is

$$\phi = \left(\pi r^2\right)\frac{\mu_0}{2\pi}\frac{Ni}{R} = \frac{\mu_0 Nir^2}{2R}$$

This is the flux that links each turn. When the current i varies with time, the induced emf e across the terminals of toroid is given by Faraday's law.

$$e = -\frac{Nd\phi}{dt} = -N\frac{d}{dt}\left(\frac{\mu_0 Nir^2}{2R}\right)$$

$$e = -N\left(\frac{\mu_0 Nr^2}{2R}\right)\frac{di}{dt}$$

Comparing with $e = -L\frac{di}{dt}$

We get,

$$L = \frac{\mu_0}{2}\frac{N^2 r^2}{R} \quad (\because r << R)$$

Given,

N=1200, r =2.0 cm, R=15 cm and

$\mu_0 = 4\pi \times 10^{-7}$ T.m/A.

$L = 2.414 \times 10^{-3}$ H

Example 12.6: Consider a uniformly wound solenoid having N turns and length l. The core of the solenoid is air. Find the inductance of the solenoid of N=200, l=20 cm and cross-sectional area, A= 5 cm^2. Calculate the induced emf e_L, if the current flowing through the solenoid decreases at a rate of 60 A/s.

Solution: The magnetic flux through each turn of area A in the solenoid is

$\phi_B = B{\cdot}A = (\mu_0 ni){\cdot}A$ ($\because$Magnetic field inside a solenoid is $B = \mu_0 ni$)

$= \mu_0 (N/l){\cdot}i{\cdot} A$($\because n$ is the number of turns per unit length=N/l)

We know that the inductance (L) of the solenoid can be written as

$L = (N\phi_B)/ i$

Substituting the value of ϕ_B, we get

$L = (N/i){\cdot} \{\mu_0 {\cdot}(N/l){\cdot}i{\cdot} A\}$

$L = \mu_0 {\cdot}(N^2/l){\cdot}A$

Inserting the given values of N, l and A, we find

$L=(4\pi \cdot 10^{-7} Tm/A)\cdot(200)^2(5\cdot 10^{-4} m^2)/(20\cdot 10^{-4} m)$

or $L \approx 0.1257$mH

The induced emf in the solenoid

$e_L = -L\,(di/dt)$

$e_L = -(0.126\cdot 10^{-3})\,(-60$ A/s$)=7.543$ mV

Example 12.7: The self-inductance of a closely wound coil of 200 turns is 10 mH. Determine the value of magnetic flux through the cross-section of the coil when the current passing through the coil is 4 mA.

Solution: Given :

Self-inductance of coil, L = 10 mH,

Number of turns, N = 200, and

Current through the coil, i = 4 mA

The total value of magnetic flux ϕ associated with the coil is,

$\phi = L\,i$

$= (10 \times 10^{-3})$ H $\times (4 \times 10^{-3})$ A

$= 4 \times 10^{-5}$ Wb

The flux per turn (or flux through the cross-section of the coil)

$$= \frac{\phi}{N}$$

$$= \left(\frac{4\times 10^{-5}\,\text{Wb}}{200}\right)$$

$= 2 \times 10^{-7}$ Wb

Inductances in series or parallel:

If several inductances are connected in series or in parallel, then the total inductance is determined by using following relations:

$$L_{\text{Total}} = L_1 + L_2 + L_3 + \ldots \quad \text{(Series Combination)}$$

$$\frac{1}{L_{Total}} = \frac{1}{L_1} + \frac{1}{L_2} + \frac{1}{L_3} + \ldots \quad \text{(Parallel Combination)}$$

12.12 Energy Stored in a Magnetic Field:

We have seen that the changing magnetic flux in a coil causes an induced emf. The induced emf so produced opposes the change and hence the energy has to be spent to overcome it to build up the magnetic field. This energy may be recovered as heat in a resistance of the circuit. This fact gives the logical concept of the energy being stored in the magnetic field.

We have dealt with a similar problem in electrostatics where the total electrostatic energy U_E is stored in the medium between the plates of a capacitor with capacitance C and charge q held at potential V is

$$U_E = \frac{q^2}{2C} = \frac{CV^2}{2} \qquad [\because q = CV]$$

Now we can estimate the energy spent to build up a current I in a circuit having an inductance L.

From Eq. (12.29),

The induced emf $e = -L\dfrac{di}{dt}$

The work alone in moving a charge dq against this emf is

$$dw = -e\cdot dq = L\frac{di}{dt}\cdot dq$$

$$= L\cdot\frac{di\cdot dq}{dt}$$

$$= L\cdot i\cdot di \qquad \left[\because \frac{dq}{dt} = i\right]$$

Therefore total work

$$W = \int dw = \int_0^I L i\,di = \frac{1}{2}Li^2 = U_B \qquad \text{--- (12.35)}$$

This is the energy stored (U_B) in magnetic field and is analogous to the energy stored (U_E) in the electric field in a capacitor given above. It can be shown that this energy stored up in magnetic field per unit volume (u_B) comes out to be ($B^2/2\,\mu_0$) Joules, which parallels the $(1/2)\varepsilon_0 E^2$, the energy density (u_E) in an electric field E, μ_0 and ε_0 being the permeability and permittivity of free space.

12.13 Energy Density of a Magnetic Field:

Consider a long solenoid having length, l near the middle, cross-sectional area $\vec{A}$ and carrying a current i through it (Fig. 12.12). The volume associated with length l will be $A.l$. The energy, U_B stored by the length l of the solenoid must lie entirely within volume Al, because the magnetic field outside the solenoid is almost zero. Moreover, the energy stored will be uniformly distributed within the volume as the magnetic field $\vec{B}$ is uniform everywhere inside the solenoid.

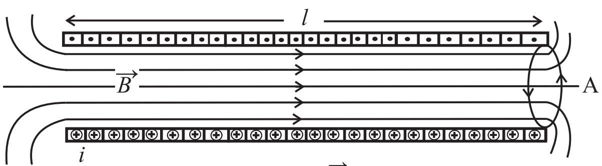


Fig 12.12 : A current carrying solenoid produces uniform magnetic field in the interior region.

Thus, the energy stored, per unit volume, in the magnetic field is

$$u_B = \frac{U_B}{A \cdot l} \qquad \text{--- (12.36)}$$

From Eq. 12.35, we know that $U_B = \frac{1}{2} LI^2$

$$\therefore u_B = \frac{1}{2} LI^2 \cdot \frac{1}{A \cdot l} = \left(\frac{L}{l}\right) \cdot \frac{I^2}{2A} \qquad \text{--- (12.37)}$$

For a long solenoid, we know that the inductance (L) per unit length is given by Eq. (12. 34) as

$$\left(\frac{L}{l}\right) = \mu_0 n^2 A,$$

where L is the inductance of a long solenoid having length l in the middle, n is the number of turns per unit length, and A is the cross-sectional area of the solenoid, μ_0 is the permeability constant for air ($4\pi \times 10^{-7}$ T.m/A or $4\pi \times 10^{-7}$ H/m) [∵1 H (Henry) = 1 T.m^2/A]

Substituting the value of (L/l) in Eq. (12.37), we get

$$\therefore u_B = \mu_0 n^2 A \cdot \frac{I^2}{2A}$$

$$u_B = \frac{1}{2} \mu_0 n^2 I^2 \qquad \text{--- (12.38)}$$

For a solenoid the magnetic field at interior points is given by Eq. (10.65) as $B = \mu_0 I.n.$

Therefore, the expression for energy density (u_B) stored in magnetic field can be written as

$$u_B = \frac{B^2}{2\mu_0} \qquad \text{--- (12.39)}$$

This equation gives the density of stored energy at any point where magnetic field is B. This equation holds good for all magnetic fields, no matter how they are produced.

Example 12.8 : Calculate the self-inductance of a coaxial cable of length l and carrying a current I. The current flows down the inner cylinder with radius a, and flows out of the outer cylinder with radius b.

Solution: According to Ampere's law, the magnetic field (B) between two cylinders at a distance r from the axis is given by

$$B = \frac{\mu_0 I}{2\pi r}.$$

The magnetic field is zero elsewhere.

We also know that the magnetic energy density,

$$u_B = \frac{B^2}{2\mu_0} = \frac{1}{2\mu_0}\left(\frac{\mu_0^2 I^2}{4\pi^2 r^2}\right) = \frac{\mu_0 I^2}{8\pi^2 r^2}$$

Energy stored in a cylindrical shell of length l, radius r and thickness dr is given by

$$\left(\frac{\mu_0 I^2}{8\pi^2 r^2}\right) \times 2\pi l r dr = \frac{\mu_0 I^2 l}{4\pi}\left(\frac{dr}{r}\right)$$

Integrating from a to b, we get

$$W = \frac{\mu_0 I^2 l}{4\pi} ln\left(\frac{b}{a}\right)$$

Magnetic energy confined in an inductor (L) carrying a current (I) can also be written as $\frac{1}{2} LI^2$. Comparing the two expressions we find the inductance of coaxial cable as

$$L = \frac{\mu_0 l}{2\pi} ln\left(\frac{b}{a}\right)$$

12.14 Mutual Inductance (M):

Let us consider a case of two coils placed side by side as shown in Fig. 12.13. Suppose a fixed current I_1 is flowing through coil 1. Due to this current a magnetic field B_1 (x,y,z) will be produced in the nearby region surrounding the coil 1. Let ϕ_{21} be the magnetic flux liked with the surface area s_2 of the coil 2 due to magnetic field $\vec{B}_1$ and can be written as

$$\phi_{21} = \int_{s_2} \vec{B}_1 \cdot \overrightarrow{\delta a}, \quad \text{---- (12.40)}$$

where s_2 represents the effective surface (or area) enclosed by coil 2. If the positions of the coils are fixed in space,

Then $\phi_{21} \propto I_1$

ϕ_{21} = constant. I_1

or $\phi_{21} = M_{21} I_1$ --- (12.41)

where, M_{21} is a constant of proportionality and is termed as mutual inductance or coefficient of mutual induction of coil 2 (or circuit c_2) with respect to coil 1 (or circuit c_1). Suppose I_1 changes slowly with time then magnetic field B_1 in the vicinity of coil 2 is related to current I_1 in coil 1 in the same way as it would be related for a steady current. The magnetic flux ϕ_{21} will change in proportion as I_1 changes.

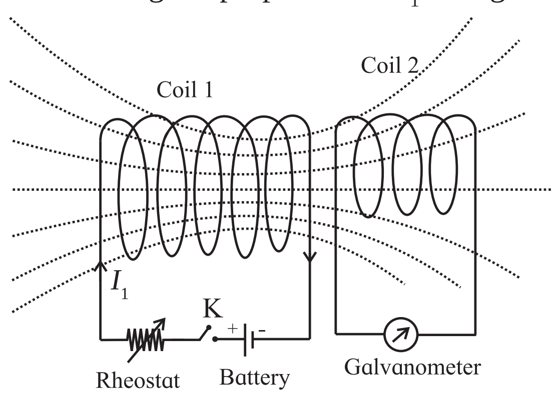


Fig. 12.13: Mutual inductance of two coils.

The induced emf in coil 2 will be written as

$$e_{21} = -\frac{d\phi_{21}}{dt}$$

$$e_{21} = -M_{21}\frac{dI_1}{dt}$$

Now we allow current I_2 to flow through coil 2. On account of this current, magnetic flux ϕ_{12} liked with coil 1 is obviously proportional to I_2.

That is

$\phi_{12} \propto I_2$

or $\phi_{12} = M_{12} I_2$ --- (12.42)

or $M_{12} = \dfrac{\phi_{12}}{I_2}$ --- (12.43)

M_{12} is known as mutual inductance of coil 1 with respect to coil 2. The induced emf in coil 1 will be

$$e_{12} = -M_{12}\frac{dI_2}{dt} \quad \text{--- (12.44)}$$

It may be noted that by symmetry, $M_{12} = M_{21} = M$.

Alternative definitions of mutual inductance:

It is evident from the Eq. (12.41) and Eq. (12.42) that

$\phi_{21} = MI_1$ and $\phi_{12} = MI_2$

or $M = \dfrac{\phi_{21}}{I_1} = \dfrac{\phi_{12}}{I_2}$ --- (12.45)

Hence, the mutual inductance of two circuits is equal to the magnetic flux linked with one circuit per unit current in the other circuit. The circuit in which current is provided by an external source is usually referred to as primary circuit while the other as secondary.

Therefore, the mutual inductance M of two circuits (or coils) is the magnetic flux (ϕ_s) linked with the secondary circuit per unit current (I_p) of the primary circuit.

$$\therefore M = \frac{\phi_s}{I_p}$$

or $\phi_s = MI_p$ --- (12.46)

Also from Faraday's law

$$e_s = -\frac{d\phi_s}{dt} = -\frac{d}{dt}(MI_p) = -M\frac{dI_p}{dt}$$

or $M = \left|\dfrac{e_s}{(dI_p/dt)}\right|$ --- (12.47)

Hence, mutual inductance is defined as the value of induced emf produced in the secondary circuit per unit rate of change in current in the primary circuit.

Use your brain power

It can be shown that the mutual potential energy of two circuits is $W = MI_1I_2$. Therefore, the mutual inductance (M) may also be defined as the mutual potential energy (W) of two circuits corresponding to unit current flowing in each circuit.

$$M = \frac{W}{I_1 I_2}$$

$M = W\ [I_1 = I_2 = 1]$

The unit of mutual inductance is henry (H).

$$\text{henry} = \frac{\text{volt}}{\text{As}} = \text{ohm} \cdot \text{s}$$

1 henry = 1 ohm.s

If corresponding to 1 A/s rate of change of current in the primary circuit, the induced emf produced in the secondary circuit is 1 volt, then the mutual inductance (*M*) of the two circuits is 1 H.

Example 12.9 : Mutual inductance of the wireless charging system.

In a wireless battery charger, the base unit can be imagined as a solenoid (coil B) of length l with N_B turns, carrying a current i_B and having a cross-section area *A*. The handle coil (coil *H*) has N_H turns and surrounds the base solenoid (coil B) completely. The base unit is designed to hold the handle of the charging unit. The handle has a cylindrical hole so that it fits loosely over a matching cylinder on the base unit. When the handle is placed on the base, the current flowing in coil *B* induces a current in the coil *H*. Thus, the induced current in the coil *H* is used to charge the battery housed in the handle.

The magnetic field due to a solenoid coil B,

$$B_{\text{solenoid}} = \mu_0 n i = \mu_0 \left(\frac{N_B}{l}\right) i$$

Magnetic flux through coil *H* caused by the magnetic field $B_{solenoid}$ due to solenoid coil *B*,

$$\phi_H = B_{solenoid} A$$

Flux linkage = $N_H \phi_H$

The mutual inductance (*M*) of the wireless charging system,

$$M = \frac{N_H B_{\text{solenoid}} A}{i} = \mu_0 \left(\frac{N_B}{l}\right) A . N_H$$

$$= \mu_0 \left(\frac{N_B N_H}{l}\right) A$$

Coefficient of coupling between two circuits:

The coefficient of coupling (*K*) is a measure of the portion of flux that reaches coil 2 which is in the vicinity of coil 1. The greater is the coefficient of coupling the greater will be the mutual inductance (M).

Inductance of any circuit is proportional to the induced voltage it can develop. This is equally true for mutual inductance.

$M \propto e_{21}$, where e_{21} is induced emf developed in coil 2 due to the portion of the flux from coil 1 reaching coil 2 (= $K\phi_1$).

But induced emf is also proportional to the number of turns in the coil,

So, $e_{21} \propto N_2 (K\phi_1)$

But $\phi_1 \propto N_1$

$\therefore e_{21} \propto N_2 (KN_1)$

Also $L \propto N^2$ or $N \propto \sqrt{L}$

$\therefore N_1 N_2 \propto \sqrt{L_1}\sqrt{L_2} = \sqrt{L_1 L_2}$

Replacing e_{21} with M, we now have

$$M = K\sqrt{L_1 L_2} \quad \text{--- (12.48)}$$

K is usually less than unity. If *K* =1, the two coils will be perfectly coupled, and $M = \sqrt{L_1 L_2}$.

(i) If $K > 0.5$, the two coils are tightly coupled

(ii) If $K<0.5$, the coils are loosely coupled.

(iii) If $L_1 = L_2$, then a coil with self-inductance *L* is coupled to itself with mutual inductance

$$M = \sqrt{L_1 L_2} = \sqrt{L^2} = L$$

It may not be always desirable to have a large value of mutual inductance (M). A large value of M is desirable for a transformer but higher M is not desirable for home appliances such as a electric clothes dryer. A dangerous emf can be induced on the metallic case of the dryer if the mutual inductance between its heating coils and the case is large. In order to minimise *M* the heating coils are counter wound so that their magnetic fields cancel one another and reduces *M* with the case of the dryer.

Theoretically, the coupling between two coils is never perfect. If two coils are wound on a common iron core, the coefficient of coupling (*K*) can be considered as unity. For two air-core coils or two coils on separate iron cores, the coefficient of coupling depends on the distance between two coils and the angle between the axes of the two coils. When the coils are parallel (and in line), the coefficient K is maximum. If the axes of the coils are at right angles (and in line), K is minimum. If we want to prevent interaction between the coils, the coils should be oriented at right angle to each other and be kept as far apart as possible. K-value for radio coils (Radio frequency, intermediate frequency transformers) lies between 0.001 to 0.05.

Use your brain power

Prove that the inductance of parallel wires of length l in the same circuit is given by $L = \left(\frac{\mu_0 l}{\pi}\right) \ln(d/a)$, where a is the radius of wire and d is separation between wire axes.

Example12.10: Two coils having self inductances L_1 = 75 mH and L_2 = 55 mH are coupled with each other. The coefficient of coupling (K) is 0.75 calculate the mutual inductance (M) of the two coils.

Solution : Given :

L_1 = 75 mH, L_2 = 55 mH, K = 0.75.

We know that,

$$M = K\sqrt{L_1 L_2}$$

$$= 0.75\sqrt{75 \times 55}\ \text{mH}$$

$$M = 48.18\ \text{mH}$$

Example 12.11: The mutual inductance (M) of the two coils is given as 1.5 H. The self inductances of the coils are :

L_1 = 5 H, L_2 = 4 H. Find the coefficient of coupling beween the coils.

Solution:

Given $L_1 = 5$ H

$L_2 = 4$ H

$M = 1.5$ H.

$$K = \frac{M}{\sqrt{L_1 L_2}} = \frac{1.5}{\sqrt{5 \times 4}}$$

$$= 0.335 = 33.55\%$$

12.15 Transformer:

Mutual inductance, is the basis of all types of transformers. A transformer is a device used for changing the voltage of alternating current from low value to high value or vice versa. We can see the transformers by road sides in villages and cities.

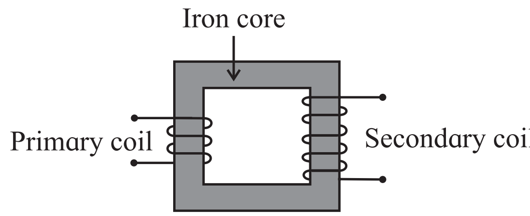


Fig. 12.14: Transformer consisting of primary and secondary coils wound on a soft iron core.

Whenever the magnetic flux linked with a coil changes, an emf is induced in the neighbouring coil. In a transformer there are two coils, primary (p) and secondary (s) insulated from each other and wound on a soft iron core as shown in Fig. 12.14. Primary and secondary coils are called input and output coils respectively.

When an AC voltage is applied to the primary coil, the current through the coil changes sinusoidally causing similar changes in the magnetic flux through the core. As the changing magnetic flux is liked with both primary and secondary coils, emf is induced in each coil. The magnetic flux linked with the coil depends upon the number of turns in the coil.

Let ϕ be the magnetic flux linked per turn with both the coils at an instant t. N_p and Ns be the number of turns in the primary and secondary coil respectively.

Then at the instant t, the magnetic flux linked with primary coil $\phi_p = Np\phi$, and with secondary coil $\phi_s = Ns\ \phi$.

The induced emf in primary and secondary coil will be

$$e_p = -\frac{d\phi_p}{dt} = -N_p \frac{d\phi}{dt}$$

and $$e_s = -N_s \frac{d\phi}{dt}$$

$$\therefore \frac{e_s}{e_p} = \frac{N_s}{N_p} \quad \text{--- (12.49)}$$

The ratio N_s/N_p is called turn ratio (transformer ratio) of the transformer. Equation (12.49) is known as equation for transformer.

For an ideal transformer,

input power = Output power

$e_p i_p = e_s i_s$

$$\frac{e_s}{e_p} = \frac{i_p}{i_s} \quad \text{--- (12.50)}$$

Combining Eqs. (12.49) and (12.50)

$$\frac{e_s}{e_p} = \frac{N_s}{N_p} = \frac{i_p}{i_s} \quad \text{--- (12.51)}$$

Case 1: When $N_s > N_p$

then $e_s > e_p$ (step up transformer)

and $i_p > i_s$. Current in the primary coil is more than that in the secondary coil.

Case 2: When $N_s < N_p$

then $e_s < e_p$ (step down transformer)

and $i_p < i_s$. Current in primary coil is less than that in secondary coil

Do you know?

Faraday's laws have found innumerable applications in modern world. Some common examples are: Electric Guitar hard drives, Smart cards, Microphones, etc. Hybrid cars: In modern days, the electric and hybrid vehicles take advantage of electromagnetic induction. The limitation of such vehicles is the life- time of a battery which is not long enough to get similar drive from a full tank of fuel/ petrol. In order to increase the amount of charge in the battery, the car acts as a generator whenever it is applying the brakes. At the time of braking, the frictional force between the tyres and the ground provides the necessary torque to the magnets inside the generator. Thus, the car takes advantage of back emf which helps in charging the battery and consequently leads to a longer drive.

Do you know?

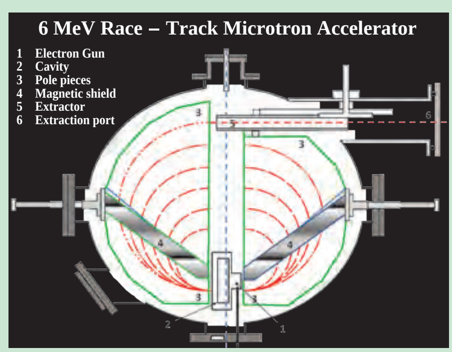
1. The flux rule is the terminology that Feynman used to refer to the law relating magnetic flux to emf. (RP Feynman, Feynman lectures on Physics, Vol II)
2. The Faraday's law relating flux to emf is referred to by Griffiths as the 'Universal flux rule'. Griffiths used the term 'Faraday's law to refer to what he called- Maxwell-Faraday equation. (DJ Griffiths, Introduction to electrodynamics 3rd Ed)

Internet my friend

https://en.wikipedia.org/wiki/Electromagnetic_induction

Do you know?

Accelerator in India:
Microtron Accelerator for electrons at Savitribai Phule Pune University



Picture credit: Dr. S.D. Dhole Department of Physics SPPU.

Exercises

1. Choose the correct option.

i) A circular coil of 100 turns with a cross-sectional area (A) of 1 m^2 is kept with its plane perpendicular to the magnetic field (B) of 1 T. What is the magnetic flux linkage with the coil?

(A) 1 Wb (B) 100 Wb

(C) 50 Wb (D) 200 Wb

ii) A conductor rod of length (l) is moving with velocity (v) in a direction normal to a uniform magnetic field (B). What will be the magnitude of induced emf produced between the ends of the moving conductor?

(A) BLv (B) BLv^2

(C) $\frac{1}{2}Blv$ (D) $\frac{2Bl}{v}$

iii) Two inductor coils with inductance 10 mH and 20 mH are connected in series. What is the resultant inductance of the combination of the two coils?

(A) 20 mH (B) 30 mH

(C) 10 mH (D) $\frac{20}{3}$ mH

iv) A current through a coil of self inductance 10 mH increases from 0 to 1 A in 0.1 s. What is the induced emf in the coil?

(A) 0.1 V (B) 1 V

(C) 10 V (D) 0.01 V

v) What is the energy required to build up a current of 1 A in an inductor of 20 mH?

(A) 10 mJ (B) 20 mJ

(C) 20 J (D) 10 J

2. Answer in brief.

i) What do you mean by electromagnetic induction? State Faraday's law of induction.

ii) State and explain Lenz's law in the light of principle of conservation of energy.

iii) What are eddy currents? State applications of eddy currents.

iv) If the copper disc of a pendulum swings between the poles of a magnet, the pendulum comes to rest very quickly. Explain the reason. What happens to the mechanical energy of the pendulum?

v) Explain why the inductance of two coils connected in parallel is less than the inductance of either coil.

3. In a Faraday disc dynamo, a metal disc of radius R rotates with an angular velocity ω about an axis perpendicular to the plane of the disc and passing through its centre. The disc is placed in a magnetic field B acting perpendicular to the plane of the disc. Determine the induced emf between the rim and the axis of the disc.

[Ans: $\frac{1}{2}(B\omega R^2)$]

4. A horizontal wire 20 m long extending from east to west is falling with a velocity of 10 m/s normal to the Earth's magnetic field of 0.5×10^{-4} T. What is the value of induced emf in the wire?

[Ans: 10 mV]

5. A metal disc is made to spin at 20 revolutions per second about an axis passing through its centre and normal to its plane. The disc has a radius of 30 cm and spins in a uniform magnetic field of 0.20 T, which is parallel to the axis of rotation. Calculate

(a) The area swept out per second by the radius of the disc,

(b) The flux cut per second by a radius of the disc,

(c) The induced emf in the disc.

[Ans: (a) 5.656 m^2, (b) 1.130 Wb, (c) 1.130V]

6. A pair of adjacent coils has a mutual inductance of 1.5 H. If the current in one coil changes from 0 to 10 A in 0.2 s, what is the change of flux linkage with the other coil?

[Ans: $d\phi$ = 15Wb, e = 75 V]

7. A long solenoid has 1500 turns/m. A coil C having cross sectional area 25 cm^2 and 150 turns (N_c) is wound tightly around the centre of the solenoid. If a current of 3.0A flows through the solenoid, calculate :

(a) the magnetic flux density at the centre of the solenoid,

(b) the flux linkage in the coil C,

(c) the average emf induced in coil C if the direction of the current in the solenoid is reversed in a time of 0.5 s.

(μ_0 = 4π × 10^{-7} *T.m*/A)

[Ans: (a) 5.66×10^{-3} T, (b) 2.12×10^{-3} Wb, (c) 8.48×10^{-3} V]

8. A search coil having 2000 turns with area 1.5 cm^2 is placed in a magnetic field of 0.60T. The coil is moved rapidly out of the field in a time of 0.2 second. Calculate the induced emf across the search coil.

[Ans: 0.9 V]

9. An aircraft of wing span of 50 m flies horizontally in earth's magnetic field of 6×10^{-5} T at a speed of 400 m/s. Calculate the emf generated between the tips of the wings of the aircraft.

[Ans: 1.2 V]

10. A stiff semi-circular wire of radius R is rotated in a uniform magnetic field B about an axis passing through its ends. If the frequency of rotation of the wire is *f*, calculate the amplitude of the alternating emf induced in the wire.

[Ans: $e_0=\pi^2BR^2f$]

11. Calculate the value of induced emf between the ends of an axle of a railway carriage 1.75 m long traveling on level ground with a uniform velocity of 50 km per hour. The vertical component of Earth's magnetic field (B_v) is given to be 5×10^{-5}T.

[Ans: 1.215 mV]

12. The value of mutual inductance of two coils is 10 mH. If the current in one of the coil changes from 5A to 1A in 0.2 s, calculate the value of emf induced in the other coil.

[Ans: e = 2 V]

13. An emf of 96.0 mV is induced in the windings of a coil when the current in a nearby coil is increasing at the rate of 1.20 A/s. What is the mutual inductance (M) of the two coils?

[Ans: 80 mH]

14. A long solenoid of length *l*, cross-sectional area *A* and having N_1 turns (primary coil), has a small coil of N_2 turns (secondary coil) wound about its centre. Determine the Mutual inductance (*M*) of the two coils.

[Ans: $M = \mu_0 N_1 N_2 A/l$]

15. The primary and secondary coil of a transformer each have an inductance of 200 ×10^{-6}H. The mutual inductance (M) between the windings is 4×10^{-6} H. What percentage of the flux from one coil reaches the other?

[Ans: 2%]

16. A toroidal ring, having 100 turns per cm of a thin wire is wound on a non-magnetic metal rod of length 1 m and diameter 1 cm. If the permeability of bar is equal to that of free space (μ_0), calculate the magnetic field inside the bar (*B*) when the current (*i*) circulating through the turns is 1 A. Also determine the self-inductance (*L*) of the coil.

[Ans: 1.256×10^{-2}T, 9.872 mH]

17. A uniform magnetic field *B(t)*, pointing upward fills a circular region of radius, s in horizontal plane. If *B* is changing with time, find the induced electric field.

[Ans: $-\pi s^2 \frac{dB}{dt}$]

[Hint : Part of Maxwell's equation, applied to a time varying magnetic flux, leads us to the equation $\oint \vec{E}\cdot\vec{dl} = \frac{-d\phi_m}{dt}$,where $\vec{E}$ is the electric field induced when the magnetic flux changes at the rate of $\frac{d\phi_m}{dt}$]

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