

1. DIFFERENTIATION



Let us Study

- Derivatives of Composite functions.
- Derivatives of Inverse functions
- Derivatives of Implicit functions.
- Higher order Derivatives.
- Geometrical meaning of Derivative.
- Logarithmic Differentiation
- Derivatives of Parametric functions.



Let us Recall

- The derivative of $f(x)$ with respect to x , at $x = a$ is given by $f'(a) = \lim_{h \rightarrow 0} \left[\frac{f(a+h) - f(a)}{h} \right]$
- The derivative can also be defined for $f(x)$ at any point x on the open interval as $f'(x) = \lim_{h \rightarrow 0} \left[\frac{f(x+h) - f(x)}{h} \right]$. If the function is given as $y = f(x)$ then its derivative is written as $\frac{dy}{dx} = f'(x)$.
- For a differentiable function $y = f(x)$ if δx is a small increment in x and the corresponding increment in y is δy then $\lim_{\delta x \rightarrow 0} \left(\frac{\delta y}{\delta x} \right) = \frac{dy}{dx}$.
- Derivatives of some standard functions.

$y = f(x)$	$\frac{dy}{dx} = f'(x)$
c (Constant)	0
x^n	nx^{n-1}
$\frac{1}{x}$	$-\frac{1}{x^2}$
$\frac{1}{x^n}$	$-\frac{n}{x^{n+1}}$
\sqrt{x}	$\frac{1}{2\sqrt{x}}$
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$

$y = f(x)$	$\frac{dy}{dx} = f'(x)$
$\sec x$	$\sec x \tan x$
$\operatorname{cosec} x$	$-\operatorname{cosec} x \cot x$
$\cot x$	$-\operatorname{cosec}^2 x$
e^x	e^x
a^x	$a^x \log a$
$\log x$	$\frac{1}{x}$
$\log_a x$	$\frac{1}{x \log a}$

Table 1.1.1

Rules of Differentiation :

If u and v are differentiable functions of x such that

$$(i) \quad y = u \pm v \text{ then } \frac{dy}{dx} = \frac{du}{dx} \pm \frac{dv}{dx} \qquad (ii) \quad y = uv \text{ then } \frac{dy}{dx} = u \frac{dv}{dx} + v \frac{du}{dx}$$

$$(iii) \quad y = \frac{u}{v} \text{ where } v \neq 0 \text{ then } \frac{dy}{dx} = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2}$$

Introduction :

The history of mathematics presents the development of calculus as being accredited to Sir Isaac Newton (1642-1727) an English physicist and mathematician and Gottfried Wilhelm Leibnitz (1646-1716) a German physicist and mathematician. The Derivative is one of the fundamental ideas of calculus. It's all about rate of change in a function. We try to find interpretations of these changes in a mathematical way. The symbol δ will be used to represent the change, for example δx represents a small change in the variable x and it is read as "change in x " or "increment in x ". δy is the corresponding change in y if y is a function of x .

We have already studied the basic concept, derivatives of standard functions and rules of differentiation in previous standard. This year, in this chapter we are going to study the geometrical meaning of derivative, derivatives of Composite, Inverse, Logarithmic, Implicit and Parametric functions and also higher order derivatives. We also add some more rules of differentiation.



Let us Learn

1.1.1 Derivatives of Composite Functions (Function of another function) :

So far we have studied the derivatives of simple functions like $\sin x$, $\log x$, e^x etc. But how about the derivatives of $\sin \sqrt{x}$, $\log(\sin(x^2 + 5))$ or $e^{\tan x}$ etc ? These are known as composite functions. In this section let us study how to differentiate composite functions.

1.1.2 Theorem : If $y = f(u)$ is a differentiable function of u and $u = g(x)$ is a differentiable function of x such that the composite function $y = f[g(x)]$ is a differentiable function of x then $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$.

Proof : Given that $y = f(u)$ and $u = g(x)$. We assume that u is not a constant function. Let there be a small increment in the value of x say δx then δu and δy are the corresponding increments in u and y respectively.

As δx , δu , δy are small increments in x , u and y respectively such that $\delta x \neq 0$, $\delta u \neq 0$ and $\delta y \neq 0$.

$$\text{We have } \frac{\delta y}{\delta x} = \frac{\delta y}{\delta u} \times \frac{\delta u}{\delta x}.$$

Taking the limit as $\delta x \rightarrow 0$ on both sides we get,



$$\lim_{\delta x \rightarrow 0} \left(\frac{\delta y}{\delta x} \right) = \lim_{\delta x \rightarrow 0} \left(\frac{\delta y}{\delta u} \right) \times \lim_{\delta x \rightarrow 0} \left(\frac{\delta u}{\delta x} \right)$$

As $\delta x \rightarrow 0$, we get, $\delta u \rightarrow 0$ ($\because u$ is a continuous function of x)

$$\lim_{\delta x \rightarrow 0} \left(\frac{\delta y}{\delta x} \right) = \lim_{\delta u \rightarrow 0} \left(\frac{\delta y}{\delta u} \right) \times \lim_{\delta x \rightarrow 0} \left(\frac{\delta u}{\delta x} \right) \quad \dots \dots \text{(I)}$$

Since y is a differentiable function of u and u is a differentiable function of x . we have,

$$\lim_{\delta u \rightarrow 0} \left(\frac{\delta y}{\delta u} \right) = \frac{dy}{du} \quad \text{and} \quad \lim_{\delta x \rightarrow 0} \left(\frac{\delta u}{\delta x} \right) = \frac{du}{dx} \quad \dots \dots \text{(II)}$$

From (I) and (II), we get

$$\lim_{\delta x \rightarrow 0} \left(\frac{\delta y}{\delta x} \right) = \frac{dy}{du} \times \frac{du}{dx} \quad \dots \dots \text{(III)}$$

The R.H.S. of (III) exists and is finite, implies L.H.S. of (III) also exists and is finite

$$\lim_{\delta x \rightarrow 0} \left(\frac{\delta y}{\delta x} \right) = \frac{dy}{dx}. \quad \text{Then equation (III) becomes,}$$

$$\boxed{\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}}$$

Note:

1. The derivative of a composite function can also be expressed as follows. $y = f(u)$ is a differentiable function of u and $u = g(x)$ is a differentiable function of x such that the composite function $y = f[g(x)]$ is defined then

$$\frac{dy}{dx} = f'[g(x)] \cdot g'(x).$$

2. If $y = f(v)$ is a differentiable function of v and $v = g(u)$ is a differentiable function of u and $u = h(x)$ is a differentiable function of x then

$$\frac{dy}{dx} = \frac{dy}{dv} \times \frac{dv}{du} \times \frac{du}{dx}.$$

3. If y is a differentiable function of u_1 , u_i is a differentiable function of u_{i+1} for $i = 1, 2, \dots, n-1$ and u_n is a differentiable function of x , then

$$\frac{dy}{dx} = \frac{dy}{du_1} \times \frac{du_1}{du_2} \times \frac{du_2}{du_3} \times \dots \times \frac{du_{n-1}}{du_n} \times \frac{du_n}{dx}$$

This rule is also known as **Chain rule**.

1.1.3 Derivatives of some standard Composite Functions :

y	$\frac{dy}{dx}$
$[f(x)]^n$	$n [f(x)]^{n-1} \cdot f'(x)$
$\sqrt{f(x)}$	$\frac{f'(x)}{2\sqrt{f(x)}}$
$\frac{1}{[f(x)]^n}$	$-\frac{n \cdot f'(x)}{[f(x)]^{n+1}}$
$\sin [f(x)]$	$\cos [f(x)] \cdot f'(x)$
$\cos [f(x)]$	$-\sin [f(x)] \cdot f'(x)$
$\tan [f(x)]$	$\sec^2 [f(x)] \cdot f'(x)$
$\sec [f(x)]$	$\sec [f(x)] \cdot \tan [f(x)] \cdot f'(x)$

y	$\frac{dy}{dx}$
$\cot [f(x)]$	$-\operatorname{cosec}^2 [f(x)] \cdot f'(x)$
$\operatorname{cosec} [f(x)]$	$-\operatorname{cosec} [f(x)] \cdot \cot [f(x)] \cdot f'(x)$
$a^{f(x)}$	$a^{f(x)} \cdot \log a \cdot f'(x)$
$e^{f(x)}$	$e^{f(x)} \cdot f'(x)$
$\log [f(x)]$	$\frac{f'(x)}{f(x)}$
$\log_a [f(x)]$	$\frac{f'(x)}{f(x) \log a}$

Table 1.1.2

SOLVED EXAMPLES

Ex. 1 : Differentiate the following w. r. t. x .

(i) $y = \sqrt{x^2 + 5}$

(ii) $y = \sin (\log x)$

(iii) $y = e^{\tan x}$

(iv) $\log (x^5 + 4)$

(v) $5^{3 \cos x - 2}$

(vi) $y = \frac{3}{(2x^2 - 7)^5}$

Solution : (i) $y = \sqrt{x^2 + 5}$

Method 1 :

Let $u = x^2 + 5$ then $y = \sqrt{u}$, where y is a differentiable function of u and u is a differentiable function of x then

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} \quad \dots \dots (I)$$

Now, $y = \sqrt{u}$

Differentiate w. r. t. u

$$\frac{dy}{du} = \frac{d}{du}(\sqrt{u}) = \frac{1}{2\sqrt{u}} \text{ and } u = x^2 + 5$$

Differentiate w. r. t. x

$$\frac{du}{dx} = \frac{d}{dx}(x^2 + 5) = 2x$$

Now, equation (I) becomes,

$$\frac{dy}{dx} = \frac{1}{2\sqrt{u}} \times 2x = \frac{x}{\sqrt{x^2 + 5}}$$

Method 2 :

We have $y = \sqrt{x^2 + 5}$

Differentiate w. r. t. x

$$\frac{dy}{dx} = \frac{d}{dx}(\sqrt{x^2 + 5})$$

[Treat $x^2 + 5$ as u in mind and use the formula of derivative of \sqrt{u}]

$$\frac{dy}{dx} = \frac{1}{2\sqrt{x^2 + 5}} \cdot \frac{d}{dx}(x^2 + 5)$$

$$\frac{dy}{dx} = \frac{1}{2\sqrt{x^2 + 5}}(2x)$$

$$\frac{dy}{dx} = \frac{x}{\sqrt{x^2 + 5}}$$

Ex. 2 : Differentiate the following w. r. t. x .

(i) $y = \sqrt{\sin x^3}$

(ii) $y = \cot^2(x^3)$

(iii) $y = \log [\cos(x^5)]$

(iv) $y = (x^3 + 2x - 3)^4(x + \cos x)^3$

(v) $y = (1 + \cos^2 x)^4 \times \sqrt{x + \sqrt{\tan x}}$

Solution :

(i) $y = \sqrt{\sin x^3}$

Differentiate w. r. t. x

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx}(\sqrt{\sin x^3}) \\ &= \frac{1}{2\sqrt{\sin x^3}} \times \frac{d}{dx}(\sin x^3) \\ &= \frac{1}{2\sqrt{\sin x^3}} \times \cos x^3 \times \frac{d}{dx}(x^3) \\ &= \frac{1}{2\sqrt{\sin x^3}} \times \cos x^3 \times (3x^2) \end{aligned}$$

$$\therefore \frac{dy}{dx} = \frac{3x^2 \cos x^3}{2\sqrt{\sin x^3}}$$

(ii) $y = \cot^2(x^3)$

Differentiate w. r. t. x

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx}(\cot^2(x^3)) \\ &= \frac{d}{dx}[\cot(x^3)]^2 \\ &= 2 \cot(x^3) \frac{d}{dx}[\cot(x^3)] \\ &= 2 \cot(x^3)[- \operatorname{cosec}^2(x^3)] \frac{d}{dx}(x^3) \\ &= -2 \cot(x^3) \operatorname{cosec}^2(x^3)(3x^2) \end{aligned}$$

$$\therefore \frac{dy}{dx} = -6x^2 \cot(x^3) \operatorname{cosec}^2(x^3)$$

(iii) $y = \log [\cos(x^5)]$

Differentiate w. r. t. x

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx}(\log [\cos(x^5)]) \\ &= \frac{1}{\cos(x^5)} \cdot \frac{d}{dx}[\cos(x^5)] \\ &= \frac{1}{\cos(x^5)} (-\sin(x^5)) \frac{d}{dx}(x^5) \end{aligned}$$

$$\therefore \frac{dy}{dx} = -\tan(x^5)(5x^4) = -5x^4 \tan(x^5)$$

(iv) $y = (x^3 + 2x - 3)^4(x + \cos x)^3$

Differentiate w. r. t. x

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx}[(x^3 + 2x - 3)^4(x + \cos x)^3] \\ &= (x^3 + 2x - 3)^4 \cdot \frac{d}{dx}(x + \cos x)^3 + (x + \cos x)^3 \cdot \frac{d}{dx}(x^3 + 2x - 3)^4 \end{aligned}$$

Solution :

(i) $y = \log_3 (\log_5 x)$

$$= \log_3 \left(\frac{\log x}{\log 5} \right) = \log_3 (\log x) - \log_3 (\log 5)$$

$$\therefore y = \frac{\log(\log x)}{\log 3} - \log_3 (\log 5)$$

Differentiate w. r. t. x

$$\frac{dy}{dx} = \frac{d}{dx} \left[\frac{\log(\log x)}{\log 3} - \log_3 (\log 5) \right]$$

$$= \frac{1}{\log 3} \frac{d}{dx} [\log(\log x)] - \frac{d}{dx} [\log_3 (\log 5)]$$

$$= \frac{1}{\log 3} \times \frac{1}{\log x} \frac{d}{dx} (\log x) - 0 \quad [\text{Note that } \log_3 (\log 5) \text{ is constant}]$$

$$= \frac{1}{\log 3} \times \frac{1}{\log x} \times \frac{1}{x}$$

$$\therefore \frac{dy}{dx} = \frac{1}{x \log x \log 3}$$

(ii) $y = \log \left[e^{3x} \cdot \frac{(3x-4)^{\frac{2}{3}}}{\sqrt[3]{2x+5}} \right] = \log \left[\frac{e^{3x} \cdot (3x-4)^{\frac{2}{3}}}{(2x+5)^{\frac{1}{3}}} \right]$

$$= \log \left[e^{3x} \cdot (3x-4)^{\frac{2}{3}} \right] - \log \left[(2x+5)^{\frac{1}{3}} \right]$$

$$= \log e^{3x} + \log(3x-4)^{\frac{2}{3}} - \log(2x+5)^{\frac{1}{3}}$$

$$\therefore y = 3x + \frac{2}{3} \log(3x-4) - \frac{1}{3} \log(2x+5) \quad [\because \log e = 1]$$

Differentiate w. r. t. x

$$\frac{dy}{dx} = \frac{d}{dx} \left[3x + \frac{2}{3} \log(3x-4) - \frac{1}{3} \log(2x+5) \right]$$

$$= 3 \frac{d}{dx} (x) + \frac{2}{3} \cdot \frac{d}{dx} [\log(3x-4)] - \frac{1}{3} \cdot \frac{d}{dx} [\log(2x+5)]$$

$$= 3(1) + \frac{2}{3} \cdot \frac{1}{3x-4} \cdot \frac{d}{dx} (3x-4) - \frac{1}{3} \cdot \frac{1}{2x+5} \cdot \frac{d}{dx} (2x+5)$$

$$= 3(1) + \frac{2}{3} \cdot \frac{1}{3x-4} \cdot (3) - \frac{1}{3} \cdot \frac{1}{2x+5} \cdot (2)$$

$$\therefore \frac{dy}{dx} = 3 + \frac{2}{3x-4} - \frac{2}{3(2x+5)}$$

$$(iii) \quad y = \log \left[\sqrt{\frac{1 - \cos\left(\frac{3x}{2}\right)}{1 + \cos\left(\frac{3x}{2}\right)}} \right] = \log \left[\sqrt{\frac{2 \sin^2\left(\frac{3x}{4}\right)}{2 \cos^2\left(\frac{3x}{4}\right)}} \right]$$

$$\therefore y = \log \left[\tan\left(\frac{3x}{4}\right) \right]$$

Differentiate w. r. t. x

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx} \left\{ \log \left[\tan\left(\frac{3x}{4}\right) \right] \right\} \\ &= \frac{1}{\tan\left(\frac{3x}{4}\right)} \cdot \frac{d}{dx} \left[\tan\left(\frac{3x}{4}\right) \right] \\ &= \cot\left(\frac{3x}{4}\right) \cdot \sec^2\left(\frac{3x}{4}\right) \cdot \frac{d}{dx} \left(\frac{3x}{4} \right) \\ &= \frac{\cos\left(\frac{3x}{4}\right)}{\sin\left(\frac{3x}{4}\right)} \times \frac{1}{\cos^2\left(\frac{3x}{4}\right)} \times \frac{3}{4} \\ &= \frac{3}{2 \left[2 \sin\left(\frac{3x}{4}\right) \cdot \cos\left(\frac{3x}{4}\right) \right]} = \frac{3}{2 \sin\left(\frac{3x}{2}\right)} \end{aligned}$$

$$\therefore \frac{dy}{dx} = \frac{3}{2} \operatorname{cosec}\left(\frac{3x}{2}\right)$$

$$(iv) \quad y = \log \left[\frac{x + \sqrt{x^2 + a^2}}{\sqrt{x^2 + a^2} - x} \right] = \log \left[\frac{\sqrt{x^2 + a^2} + x}{\sqrt{x^2 + a^2} - x} \times \frac{\sqrt{x^2 + a^2} + x}{\sqrt{x^2 + a^2} + x} \right]$$

$$= \log \left[\frac{(\sqrt{x^2 + a^2} + x)^2}{x^2 + a^2 - x^2} \right]$$

$$= \log \left[\frac{(\sqrt{x^2 + a^2} + x)^2}{a^2} \right]$$

$$= \log(\sqrt{x^2 + a^2} + x)^2 - \log(a^2)$$

$$\therefore y = 2 \log(\sqrt{x^2 + a^2} + x) - \log(a^2)$$

Differentiate w. r. t. x

$$\begin{aligned}
\frac{dy}{dx} &= \frac{d}{dx} \left[2 \log(\sqrt{x^2 + a^2} + x) - \log(a^2) \right] \\
&= 2 \frac{d}{dx} \left[\log(\sqrt{x^2 + a^2} + x) \right] - \frac{d}{dx} [\log(a^2)] \\
&= 2 \times \frac{1}{\sqrt{x^2 + a^2} + x} \cdot \frac{d}{dx} [\sqrt{x^2 + a^2} + x] - 0 \\
&= \frac{2}{\sqrt{x^2 + a^2} + x} \cdot \left[\frac{1}{2\sqrt{x^2 + a^2}} \cdot \frac{d}{dx} (x^2 + a^2) + 1 \right] \\
&= \frac{2}{\sqrt{x^2 + a^2} + x} \cdot \left[\frac{1}{2\sqrt{x^2 + a^2}} (2x) + 1 \right] \\
&= \frac{2}{\sqrt{x^2 + a^2} + x} \cdot \left[\frac{x + \sqrt{x^2 + a^2}}{\sqrt{x^2 + a^2}} \right] \\
\frac{dy}{dx} &= \frac{2}{\sqrt{x^2 + a^2}}
\end{aligned}$$

(v) $y = (4)^{\log_2(\sin x)} + (9)^{\log_3(\cos x)}$
 $= (2^2)^{\log_2(\sin x)} + (3^2)^{\log_3(\cos x)}$
 $= (2)^{2\log_2(\sin x)} + (3)^{2\log_3(\cos x)}$
 $= (2)^{\log_2(\sin^2 x)} + (3)^{\log_3(\cos^2 x)} \quad [\because a^{\log_a f(x)} = f(x)]$
 $= \sin^2 x + \cos^2 x$

$\therefore y = 1$

Differentiate w. r. t. x

$$\frac{dy}{dx} = \frac{d}{dx} (1) = 0$$

(vi) $y = a^{a^{\log_a(\cot x)}}$
 $y = a^{\cot x} \quad [\because a^{\log_a f(x)} = f(x)]$

Differentiate w. r. t. x

$$\begin{aligned}
\frac{dy}{dx} &= \frac{d}{dx} (a^{\cot x}) \\
&= a^{\cot x} \log a \cdot \frac{d}{dx} (\cot x)
\end{aligned}$$

$$= a^{\cot x} \log a (-\operatorname{cosec}^2 x)$$

$$\frac{dy}{dx} = -\operatorname{cosec}^2 x \cdot a^{\cot x} \log a$$

Ex. 4 : If $f(x) = \sqrt{7g(x) - 3}$, $g(3) = 4$ and $g'(3) = 5$, find $f'(3)$.

Solution : Given that : $f(x) = \sqrt{7g(x) - 3}$

Differentiate w. r. t. x

$$f'(x) = \frac{d}{dx} (\sqrt{7g(x) - 3}) = \frac{1}{2\sqrt{7g(x) - 3}} \frac{d}{dx} [7g(x) - 3]$$

$$\therefore f'(x) = \frac{7g'(x)}{2\sqrt{7g(x) - 3}}$$

For $x = 3$, we get

$$f'(3) = \frac{7g'(3)}{2\sqrt{7g(3) - 3}} = \frac{35}{2(5)} = \frac{7}{2} \quad [\text{Since } g(3) = 4 \text{ and } g'(3) = 5]$$

(2) Differentiate the following w.r.t. x

- (i) $\cos(x^2 + a^2)$ (ii) $\sqrt{e^{(3x+2)} + 5}$
 (iii) $\log\left[\tan\left(\frac{x}{2}\right)\right]$ (iv) $\sqrt{\tan\sqrt{x}}$
 (v) $\cot^3[\log(x^3)]$ (vi) $5^{\sin^3 x + 3}$
 (vii) $\operatorname{cosec}(\sqrt{\cos x})$ (viii) $\log[\cos(x^3 - 5)]$
 (ix) $e^{3 \sin^2 x - 2 \cos^2 x}$ (x) $\cos^2[\log(x^2 + 7)]$
 (xi) $\tan[\cos(\sin x)]$ (xii) $\sec[\tan(x^4 + 4)]$
 (xiii) $e^{\log[(\log x)^2 - \log x^2]}$ (xiv) $\sin\sqrt{\sin\sqrt{x}}$
 (xv) $\log[\sec(e^{x^2})]$ (xvi) $\log_{e^2}(\log x)$
 (xvii) $[\log[\log(\log x)]]^2$
 (xviii) $\sin^2 x^2 - \cos^2 x^2$

(3) Differentiate the following w.r.t. x

- (i) $(x^2 + 4x + 1)^3 + (x^3 - 5x - 2)^4$
 (ii) $(1 + 4x)^5 (3 + x - x^2)^8$
 (iii) $\frac{x}{\sqrt{7-3x}}$ (iv) $\frac{(x^3 - 5)^5}{(x^3 + 3)^3}$
 (v) $(1 + \sin^2 x)^2 (1 + \cos^2 x)^3$
 (vi) $\sqrt{\cos x} + \sqrt{\cos\sqrt{x}}$
 (vii) $\log(\sec 3x + \tan 3x)$ (viii) $\frac{1 + \sin x^\circ}{1 - \sin x^\circ}$
 (ix) $\cot\left(\frac{\log x}{2}\right) - \log\left(\frac{\cot x}{2}\right)$
 (x) $\frac{e^{2x} - e^{-2x}}{e^{2x} + e^{-2x}}$ (xi) $\frac{e^{\sqrt{x}} + 1}{e^{\sqrt{x}} - 1}$
 (xii) $\log[\tan^3 x \cdot \sin^4 x \cdot (x^2 + 7)^7]$
 (xiii) $\log\left(\sqrt{\frac{1 - \cos 3x}{1 + \cos 3x}}\right)$
 (xiv) $\log\left(\sqrt{\frac{1 + \cos\left(\frac{5x}{2}\right)}{1 - \cos\left(\frac{5x}{2}\right)}}\right)$

- (xv) $\log\left(\sqrt{\frac{1 - \sin x}{1 + \sin x}}\right)$
 (xvi) $\log\left[4^{2x} \left(\frac{x^2 + 5}{\sqrt{2x^3 - 4}}\right)^{\frac{3}{2}}\right]$
 (xvii) $\log\left[\frac{e^{x^2} (5 - 4x)^{\frac{3}{2}}}{\sqrt[3]{7 - 6x}}\right]$
 (xviii) $\log\left(\frac{a^{\cos x}}{(x^2 - 3)^3 \log x}\right)$
 (xix) $y = (25)^{\log_5(\sec x)} - (16)^{\log_4(\tan x)}$
 (xx) $\frac{(x^2 + 2)^4}{\sqrt{x^2 + 5}}$

(4) A table of values of f, g, f' and g' is given

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
2	1	6	-3	4
4	3	4	5	-6
6	5	2	-4	7

- (i) If $r(x) = f[g(x)]$ find $r'(2)$.
 (ii) If $R(x) = g[3 + f(x)]$ find $R'(4)$.
 (iii) If $s(x) = f[9 - f(x)]$ find $s'(4)$.
 (iv) If $S(x) = g[g(x)]$ find $S'(6)$.
 (5) Assume that $f'(3) = -1, g'(2) = 5, g(2) = 3$ and $y = f[g(x)]$ then $\left[\frac{dy}{dx}\right]_{x=2} = ?$
 (6) If $h(x) = \sqrt{4f(x) + 3g(x)}, f(1) = 4, g(1) = 3, f'(1) = 3, g'(1) = 4$ find $h'(1)$.
 (7) Find the x co-ordinates of all the points on the curve $y = \sin 2x - 2 \sin x, 0 \leq x < 2\pi$ where $\frac{dy}{dx} = 0$.

we have, $\lim_{\delta x \rightarrow 0} \left(\frac{\delta y}{\delta x} \right) = \frac{dy}{dx}$ and $\frac{dy}{dx} \neq 0$ (II)

From (I) and (II), we get

$$\lim_{\delta y \rightarrow 0} \left(\frac{\delta x}{\delta y} \right) = \frac{1}{\frac{dy}{dx}} \quad \dots \dots \text{(III)}$$

As $\frac{dy}{dx} \neq 0$, $\frac{1}{\frac{dy}{dx}}$ exists and is finite. $\therefore \lim_{\delta y \rightarrow 0} \left(\frac{\delta x}{\delta y} \right) = \frac{dx}{dy}$ exists and is finite.

Hence, from (III) $\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}}$ where $\frac{dy}{dx} \neq 0$

An alternative proof using derivatives of composite functions rule.

We know that $f^{-1} [f(x)] = x$ [Identity function]

Taking derivative on both sides we get,

$$\frac{d}{dx} [f^{-1} [f(x)]] = \frac{d}{dx} (x)$$

i.e. $(f^{-1})' [f(x)] \frac{d}{dx} [f(x)] = 1$

i.e. $(f^{-1})' [f(x)] f'(x) = 1$

$\therefore (f^{-1})' [f(x)] = \frac{1}{f'(x)}$ (I)

So, if $y = f(x)$ is a differentiable function of x and $x = f^{-1} (y)$ exists and is differentiable then

$$(f^{-1})' [f(x)] = (f^{-1})' (y) = \frac{dx}{dy} \text{ and } f'(x) = \frac{dy}{dx}$$

\therefore (I) becomes

$$\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}} \text{ where } \frac{dy}{dx} \neq 0$$

 **SOLVED EXAMPLES**

Ex. 1 : Find the derivative of the function $y = f(x)$ using the derivative of the inverse function $x = f^{-1}(y)$ in the following

(i) $y = \sqrt[3]{x+4}$

(ii) $y = \sqrt{1+\sqrt{x}}$

(iii) $y = \ln x$

Solution :

(i) $y = \sqrt[3]{x+4}$

We first find the inverse of the function $y = f(x)$, i.e. x in term of y .

$$y^3 = x + 4 \quad \therefore x = y^3 - 4 \quad \therefore x = f^{-1}(y) = y^3 - 4$$

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{1}{\frac{d}{dy}(y^3 - 4)} = \frac{1}{3y^2}$$

$$= \frac{1}{3(\sqrt[3]{x+4})^2} = \frac{1}{3\sqrt[3]{(x+4)^2}} \quad \text{for } x \neq -4$$

(ii) $y = \sqrt{1 + \sqrt{x}}$

We first find the inverse of the function $y = f(x)$, i.e. x in term of y .

$$y^2 = 1 + \sqrt{x} \text{ i.e. } \sqrt{x} = y^2 - 1, \therefore x = f^{-1}(y) = (y^2 - 1)^2$$

$$\begin{aligned} \frac{dy}{dx} &= \frac{1}{\frac{dx}{dy}} = \frac{1}{\frac{d}{dy}[(y^2 - 1)^2]} = \frac{1}{2(y^2 - 1)\frac{d}{dy}(y^2 - 1)} \\ &= \frac{1}{2(y^2 - 1)(2y)} = \frac{1}{4\sqrt{1 + \sqrt{x}} \left[(\sqrt{1 + \sqrt{x}})^2 - 1 \right]} \\ &= \frac{1}{4\sqrt{1 + \sqrt{x}}(1 + \sqrt{x} - 1)} = \frac{1}{4\sqrt{x}\sqrt{1 + \sqrt{x}}} \end{aligned}$$

(iii) $y = \log x$

We first find the inverse of the function $y = f(x)$, i.e. x in term of y .

$$y = \log x \therefore x = f^{-1}(y) = e^y$$

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{1}{\frac{d}{dy}(e^y)} = \frac{1}{e^y} = \frac{1}{e^{\ln x}} = \frac{1}{x}$$

Ex. 2 : Find the derivative of the inverse of function $y = 2x^3 - 6x$ and calculate its value at $x = -2$.

Solution : Given : $y = 2x^3 - 6x$

Diff. w. r. t. x we get,

$$\frac{dy}{dx} = 6x^2 - 6 = 6(x^2 - 1)$$

$$\text{we have, } \frac{dx}{dy} = \frac{1}{\frac{dy}{dx}}$$

$$\therefore \frac{dx}{dy} = \frac{1}{6(x^2 - 1)}$$

at $x = -2$,

$$\begin{aligned} \text{we get, } y &= 2(-2)^3 - 6(-2) \\ &= -16 + 12 = -4 \end{aligned}$$

$$\begin{aligned} \left[\frac{dx}{dy} \right]_{y=-4} &= \frac{1}{\left[\frac{dy}{dx} \right]_{x=-2}} \\ &= \frac{1}{6((-2)^2 - 1)} \\ &= \frac{1}{18} \end{aligned}$$

Ex. 3 : Let f and g be the inverse functions of each other. The following table lists a few values of f , g and f'

x	$f(x)$	$g(x)$	$f'(x)$
-4	2	1	$\frac{1}{3}$
1	-4	-2	4

find $g'(-4)$.

Solution : In order to find $g'(-4)$, we should first find an expression for $g'(x)$ for any input x . Since f and g are inverses we can use the following identity which holds for any two differentiable inverse functions.

$$\begin{aligned} g'(x) &= \frac{1}{f'[g(x)]} \quad \dots \text{ [check, how?]} \\ &\dots \text{ [Hint : } f[g(x)] = x \text{]} \end{aligned}$$

$$\begin{aligned} \therefore g'(-4) &= \frac{1}{f'[g(-4)]} \\ &= \frac{1}{f'(1)} = \frac{1}{4} \end{aligned}$$

Ex. 4 : Let $f(x) = x^5 + 2x - 3$. Find $(f^{-1})'(-3)$.

Solution : Given : $f(x) = x^5 + 2x - 3$

Diff. w. r. t. x we get,

$$f'(x) = 5x^4 + 2$$

Note that $f(x) = -3$ corresponds to $x = 0$.

$$\begin{aligned}\therefore (f^{-1})'(-3) &= \frac{1}{f'(0)} \\ &= \frac{1}{5(0) + 2} = \frac{1}{2}\end{aligned}$$

1.2.4 Derivatives of Standard Inverse trigonometric Functions :

We observe that inverse trigonometric functions are multi-valued functions and because of this, their derivatives depend on which branch of the function we are dealing with. We are not restricted to use these branches all the time. While solving the problems it is customary to select the branch of the inverse trigonometric function which is applicable to the kind of problem we are solving. We have to pay more attention towards the domain and range.

1. If $y = \sin^{-1} x$, $-1 \leq x \leq 1$, $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$ then prove that $\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$, $|x| < 1$.

Proof : Given that $y = \sin^{-1} x$, $-1 \leq x \leq 1$, $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$

$$\therefore x = \sin y \quad \dots \text{ (I)}$$

Differentiate w. r. t. y

$$\frac{dx}{dy} = \frac{d}{dy}(\sin y)$$

$$\frac{dx}{dy} = \cos y = \pm \sqrt{\cos^2 y} = \pm \sqrt{1 - \sin^2 y}$$

$$\therefore \frac{dx}{dy} = \pm \sqrt{1 - x^2} \quad \dots [\because \sin y = x]$$

But $\cos y$ is positive since y lies in 1st or 4th quadrant as $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$

$$\therefore \frac{dx}{dy} = \sqrt{1 - x^2}$$

$$\text{We have } \frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$$

$$\therefore \frac{dy}{dx} = \frac{1}{\sqrt{1 - x^2}}, |x| < 1$$

2. If $y = \cos^{-1} x$, $-1 \leq x \leq 1$, $0 \leq y \leq \pi$ then prove that $\frac{dy}{dx} = -\frac{1}{\sqrt{1-x^2}}$.

[As home work for students to prove.]

3. If $y = \cot^{-1} x$, $x \in R$, $0 < y < \pi$ then $\frac{dy}{dx} = -\frac{1}{1+x^2}$.

Proof: Given that $y = \cot^{-1} x$, $x \in R$, $0 < y < \pi$

$$\therefore x = \cot y \quad \dots (I)$$

Differentiate w. r. t. y

$$\frac{dx}{dy} = \frac{d}{dy}(\cot y)$$

$$\frac{dx}{dy} = -\operatorname{cosec}^2 y = -(1 + \cot^2 y)$$

$$\therefore \frac{dx}{dy} = -(1 + x^2) \quad \dots [\because \cot y = x]$$

$$\text{We have } \frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$$

$$\therefore \frac{dy}{dx} = \frac{1}{-(1+x^2)} \quad \therefore \frac{dy}{dx} = -\frac{1}{1+x^2}$$

4. If $y = \tan^{-1} x$, $x \in R$, $-\frac{\pi}{2} < y < \frac{\pi}{2}$ then $\frac{dy}{dx} = \frac{1}{1+x^2}$. [left as home work for students to prove.]

5. If $y = \sec^{-1} x$, such that $|x| \geq 1$ and $0 \leq y \leq \pi$, $y \neq \frac{\pi}{2}$ then

$$\frac{dy}{dx} = \frac{1}{x\sqrt{x^2-1}} \quad \text{if } x > 1$$

$$\frac{dy}{dx} = -\frac{1}{x\sqrt{x^2-1}} \quad \text{if } x < -1$$

Proof: Given that $y = \sec^{-1} x$, $|x| \geq 1$ and $0 \leq y \leq \pi$, $y \neq \frac{\pi}{2}$

$$\therefore x = \sec y \quad \dots (I)$$

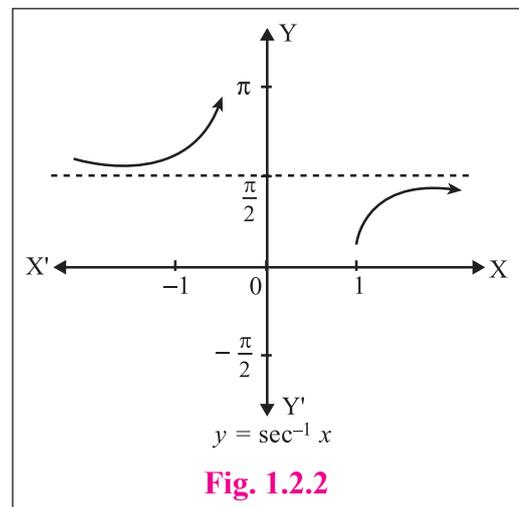
Differentiate w. r. t. y

$$\frac{dx}{dy} = \frac{d}{dy}(\sec y)$$

$$\frac{dx}{dy} = \sec y \cdot \tan y$$

$$\therefore \frac{dx}{dy} = \pm \sec y \cdot \sqrt{\tan^2 y} = \pm \sec y \cdot \sqrt{\sec^2 y - 1}$$

$$\therefore \frac{dx}{dy} = \pm x \sqrt{x^2 - 1} \quad \dots [\because \sec y = x]$$



We use the sign \pm because for y in 1st and 2nd quadrant. $\sec y \cdot \tan y > 0$.

Hence we choose $x \sqrt{x^2 - 1}$ if $x > 1$ and $-x \sqrt{x^2 - 1}$ if $x < -1$

In 1st quadrant both $\sec y$ and $\tan y$ are positive.

In 2nd quadrant both $\sec y$ and $\tan y$ are negative.

$\therefore \sec y \cdot \tan y$ is positive in both first and second quadrant.

1.2.6 Derivatives of Standard Inverse trigonometric Composite Functions :

y	$\frac{dy}{dx}$	y	$\frac{dy}{dx}$
$\sin^{-1} [f(x)]$	$\frac{f'(x)}{\sqrt{1 - [f(x)]^2}}, f(x) < 1$	$\cot^{-1} [f(x)]$	$-\frac{f'(x)}{1 + [f(x)]^2}$
$\cos^{-1} [f(x)]$	$-\frac{f'(x)}{\sqrt{1 - [f(x)]^2}}, f(x) < 1$	$\sec^{-1} [f(x)]$	$\frac{f'(x)}{f(x) \sqrt{[f(x)]^2 - 1}}, \text{ for } f(x) > 1$
$\tan^{-1} [f(x)]$	$\frac{f'(x)}{1 + [f(x)]^2}$	$\text{cosec}^{-1} [f(x)]$	$-\frac{f'(x)}{f(x) \sqrt{[f(x)]^2 - 1}}, \text{ for } f(x) > 1$

Table 1.2.2

Some Important Formulae for Inverse Trigonometric Functions :

(1) $\sin^{-1} (\sin \theta) = \theta, \sin(\sin^{-1} x) = x$	(2) $\cos^{-1} (\cos \theta) = \theta, \cos(\cos^{-1} x) = x$	
(3) $\tan^{-1} (\tan \theta) = \theta, \tan(\tan^{-1} x) = x$	(4) $\cot^{-1} (\cot \theta) = \theta, \cot(\cot^{-1} x) = x$	
(5) $\sec^{-1} (\sec \theta) = \theta, \sec(\sec^{-1} x) = x$	(6) $\text{cosec}^{-1} (\text{cosec } \theta) = \theta, \text{cosec}(\text{cosec}^{-1} x) = x$	
(7) $\sin^{-1} (\cos \theta) = \sin^{-1} \left[\sin \left(\frac{\pi}{2} - \theta \right) \right] = \frac{\pi}{2} - \theta$	(8) $\cos^{-1} (\sin \theta) = \cos^{-1} \left[\cos \left(\frac{\pi}{2} - \theta \right) \right] = \frac{\pi}{2} - \theta$	
(9) $\tan^{-1} (\cot \theta) = \tan^{-1} \left[\tan \left(\frac{\pi}{2} - \theta \right) \right] = \frac{\pi}{2} - \theta$	(10) $\cot^{-1} (\tan \theta) = \cot^{-1} \left[\cot \left(\frac{\pi}{2} - \theta \right) \right] = \frac{\pi}{2} - \theta$	
(11) $\sec^{-1} (\text{cosec } \theta) = \sec^{-1} \left[\sec \left(\frac{\pi}{2} - \theta \right) \right] = \frac{\pi}{2} - \theta$		
(12) $\text{cosec}^{-1} (\sec \theta) = \text{cosec}^{-1} \left[\text{cosec} \left(\frac{\pi}{2} - \theta \right) \right] = \frac{\pi}{2} - \theta$		
(13) $\sin^{-1} (x) = \text{cosec}^{-1} \left(\frac{1}{x} \right)$	(14) $\text{cosec}^{-1} (x) = \sin^{-1} \left(\frac{1}{x} \right)$	
(15) $\cos^{-1} (x) = \sec^{-1} \left(\frac{1}{x} \right)$	(16) $\sec^{-1} (x) = \cos^{-1} \left(\frac{1}{x} \right)$	
(17) $\tan^{-1} (x) = \cot^{-1} \left(\frac{1}{x} \right)$	(18) $\cot^{-1} (x) = \tan^{-1} \left(\frac{1}{x} \right)$	
(19) $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$	(20) $\tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}$	(21) $\sec^{-1} x + \text{cosec}^{-1} x = \frac{\pi}{2}$
(22) $\tan^{-1} x + \tan^{-1} y = \tan^{-1} \left(\frac{x+y}{1-xy} \right)$		(23) $\tan^{-1} x - \tan^{-1} y = \tan^{-1} \left(\frac{x-y}{1+xy} \right)$

In above tables, x is a real variable with restrictions. **Table 1.2.3**

Some Important Substitutions :

Expression	Substitutions	Expression	Substitutions
$\sqrt{1-x^2}$	$x = \sin \theta$ or $x = \cos \theta$	$\frac{2x}{1+x^2}$	$x = \tan \theta$
$\sqrt{1-x^2}$	$x = \tan \theta$ or $x = \cot \theta$	$\frac{1-x^2}{1+x^2}$	$x = \tan \theta$
$\sqrt{x^2+1}$	$x = \sec \theta$ or $x = \operatorname{cosec} \theta$	$\frac{3x-4x^3}{1-2x^2}$	$x = \sin \theta$
$\sqrt{\frac{a+x}{a-x}}$ or $\sqrt{\frac{a-x}{a+x}}$	$x = a \cos 2\theta$ or $x = a \cos \theta$	$\frac{4x^3-3x}{2x^2-1}$	$x = \cos \theta$
$\sqrt{\frac{1+x}{1-x}}$ or $\sqrt{\frac{1-x}{1+x}}$	$x = \cos 2\theta$ or $x = \cos \theta$	$\frac{3x-x^3}{1-3x^2}$	$x = \tan \theta$
$\sqrt{\frac{a+x^2}{a-x^2}}$ or $\sqrt{\frac{a-x^2}{a+x^2}}$	$x^2 = a \cos 2\theta$ or $x^2 = a \cos \theta$	$\frac{2f(x)}{1+[f(x)]^2}$ or $\frac{1-[f(x)]^2}{1+[f(x)]^2}$	$f(x) = \tan \theta$

Table 1.2.4



SOLVED EXAMPLES

Ex. 1 : Using derivative prove that $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$.

Solution : Let $f(x) = \sin^{-1} x + \cos^{-1} x \dots\dots$ (I)

We have to prove that $f(x) = \frac{\pi}{2}$

Differentiate (I) w. r. t. x

$$\frac{d}{dx}[f(x)] = \frac{d}{dx}[\sin^{-1} x + \cos^{-1} x]$$

$$f'(x) = \frac{1}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-x^2}} = 0$$

$f'(x) = 0 \Rightarrow f(x)$ is a constant function.

Let $f(x) = c$. For any value of x , $f(x)$ must be c only. So conveniently we can choose $x = 0$,

\therefore from (I) we get,

$$f(0) = \sin^{-1}(0) + \cos^{-1}(0) = 0 + \frac{\pi}{2} = \frac{\pi}{2} \Rightarrow c = \frac{\pi}{2} \therefore f(x) = \frac{\pi}{2}$$

$$\text{Hence, } \sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}.$$

Ex. 2 : Differentiate the following w. r. t. x .

(i) $\sin^{-1}(x^3)$

(ii) $\cos^{-1}(2x^2 - x)$

(iii) $\sin^{-1}(2^x)$

(iv) $\cot^{-1}\left(\frac{1}{x^2}\right)$

(v) $\cos^{-1}\left(\sqrt{\frac{1+x}{2}}\right)$

(vi) $\sin^2(\sin^{-1}(x^2))$

Solution :

(i) Let $y = \sin^{-1}(x^3)$

Differentiate w. r. t. x.

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}(\sin^{-1}(x^3)) \\ &= \frac{1}{\sqrt{1-(x^3)^2}} \cdot \frac{d}{dx}(x^3) \\ &= \frac{1}{\sqrt{1-x^6}} (3x^2)\end{aligned}$$

$$\therefore \frac{dy}{dx} = \frac{3x^2}{\sqrt{1-x^6}}$$

(iii) Let $y = \sin^{-1}(2^x)$

Differentiate w. r. t. x.

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}(\sin^{-1}(2^x)) \\ &= \frac{1}{\sqrt{1-(2^x)^2}} \cdot \frac{d}{dx}(2^x) \\ &= \frac{1}{\sqrt{1-2^{2x}}} (2^x \log 2)\end{aligned}$$

$$\therefore \frac{dy}{dx} = \frac{2^x \log 2}{\sqrt{1-4^x}}$$

(iv) Let $y = \cot^{-1}\left(\frac{1}{x^2}\right) = \tan^{-1}(x^2)$

Differentiate w. r. t. x.

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}(\tan^{-1}(x^2)) \\ &= \frac{1}{1+(x^2)^2} \cdot \frac{d}{dx}(x^2)\end{aligned}$$

$$\therefore \frac{dy}{dx} = \frac{2x}{1+x^4}$$

(vi) Let $y = \sin^2(\sin^{-1}(x^2))$

$$= [\sin(\sin^{-1}(x^2))]^2 = (x^2)^2$$

$$\therefore y = x^4$$

Differentiate w. r. t. x.

$$\frac{dy}{dx} = \frac{d}{dx}(x^4) \qquad \therefore \frac{dy}{dx} = 4x^3$$

(ii) Let $y = \cos^{-1}(2x^2 - x)$

$$\text{Hence } \cos y = 2x^2 - x \qquad \dots \text{ (I)}$$

Differentiate w. r. t. x.

$$-\sin y \cdot \frac{dy}{dx} = 4x - 1$$

$$\frac{dy}{dx} = \frac{1-4x}{\sin y} = \frac{1-4x}{\sqrt{1-\cos^2 y}}$$

$$\therefore \frac{dy}{dx} = \frac{1-4x}{\sqrt{1-x^2(2x-1)^2}} \qquad \dots \text{ from (I)}$$

Alternate Method :

If $y = \cos^{-1}(2x^2 - x)$

Differentiate w. r. t. x.

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}(\cos^{-1}(2x^2 - x)) \\ &= \frac{-1}{\sqrt{1-(2x^2-x)^2}} \cdot \frac{d}{dx}(2x^2 - x) \\ &= \frac{-1}{\sqrt{1-x^2(2x-1)^2}} \cdot (4x - 1)\end{aligned}$$

$$\therefore \frac{dy}{dx} = \frac{1-4x}{\sqrt{1-x^2(2x-1)^2}}$$

(v) Let $y = \cos^{-1}\left(\sqrt{\frac{1+x}{2}}\right)$

Differentiate w. r. t. x.

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx} \left[\cos^{-1} \left(\sqrt{\frac{1+x}{2}} \right) \right] \\ &= - \frac{1}{\sqrt{1-\left(\sqrt{\frac{1+x}{2}}\right)^2}} \cdot \frac{d}{dx} \left(\sqrt{\frac{1+x}{2}} \right) \\ &= - \frac{1}{\sqrt{1-\frac{1+x}{2}}} \times \frac{1}{2\sqrt{\frac{1+x}{2}}} \times \frac{d}{dx} \left(\frac{1+x}{2} \right) \\ &= - \frac{\sqrt{2}}{\sqrt{1-x}} \times \frac{1}{\sqrt{2}\sqrt{1+x}} \times \frac{1}{2}\end{aligned}$$

$$\therefore \frac{dy}{dx} = - \frac{1}{2\sqrt{1-x^2}}$$

Ex. 4 : Differentiate the following w. r. t. x.

$$(i) \sin^{-1}\left(\frac{2 \cos x + 3 \sin x}{\sqrt{13}}\right) \quad (ii) \cos^{-1}\left(\frac{3 \sin x^2 + 4 \cos x^2}{5}\right) \quad (iii) \sin^{-1}\left(\frac{a \cos x - b \sin x}{\sqrt{a^2 + b^2}}\right)$$

Solution :

$$(i) \text{ Let } y = \sin^{-1}\left(\frac{2 \cos x + 3 \sin x}{\sqrt{13}}\right)$$

$$= \sin^{-1}\left(\frac{2}{\sqrt{13}} \cos x + \frac{3}{\sqrt{13}} \sin x\right)$$

$$\text{Put } \frac{2}{\sqrt{13}} = \sin \alpha, \frac{3}{\sqrt{13}} = \cos \alpha$$

$$\text{Also, } \sin^2 \alpha + \cos^2 \alpha = \frac{4}{13} + \frac{9}{13} = 1$$

$$\text{And } \tan \alpha = \frac{2}{3} \therefore \alpha = \tan^{-1}\left(\frac{2}{3}\right)$$

$$y = \sin^{-1}(\sin \alpha \cos x + \cos \alpha \sin x)$$

$$y = \sin^{-1}(\sin x \cos \alpha + \cos x \sin \alpha)$$

$$y = \sin^{-1}[\sin(x + \alpha)]$$

$$\therefore y = x + \tan^{-1}\left(\frac{2}{3}\right)$$

Differentiate w. r. t. x.

$$\frac{dy}{dx} = \frac{d}{dx} \left[x + \tan^{-1}\left(\frac{2}{3}\right) \right] = 1 + 0$$

$$\therefore \frac{dy}{dx} = 1$$

$$(ii) \text{ Let } y = \cos^{-1}\left(\frac{3 \sin x^2 + 4 \cos x^2}{5}\right)$$

$$= \cos^{-1}\left(\frac{3}{5} \sin x^2 + \frac{4}{5} \cos x^2\right)$$

$$\text{Put } \frac{3}{5} = \sin \alpha, \frac{4}{5} = \cos \alpha$$

$$\text{Also, } \sin^2 \alpha + \cos^2 \alpha = \frac{9}{25} + \frac{16}{25} = 1$$

$$\text{And } \tan \alpha = \frac{3}{4} \therefore \alpha = \tan^{-1}\left(\frac{3}{4}\right)$$

$$y = \cos^{-1}(\sin \alpha \sin x^2 + \cos \alpha \cos x^2)$$

$$y = \cos^{-1}(\cos x^2 \cos \alpha + \sin x^2 \sin \alpha)$$

$$y = \cos^{-1}[\cos(x^2 - \alpha)]$$

$$\therefore y = x^2 - \tan^{-1}\left(\frac{3}{4}\right)$$

Differentiate w. r. t. x.

$$\frac{dy}{dx} = \frac{d}{dx} \left[x^2 - \tan^{-1}\left(\frac{3}{4}\right) \right] = 2x - 0$$

$$\therefore \frac{dy}{dx} = 2x$$

$$(iii) \text{ Let } y = \sin^{-1}\left(\frac{a \cos x - b \sin x}{\sqrt{a^2 + b^2}}\right) = \sin^{-1}\left(\frac{a}{\sqrt{a^2 + b^2}} \cos x - \frac{b}{\sqrt{a^2 + b^2}} \sin x\right)$$

$$\text{Put } \frac{a}{\sqrt{a^2 + b^2}} = \sin \alpha, \frac{b}{\sqrt{a^2 + b^2}} = \cos \alpha$$

$$\text{Also, } \sin^2 \alpha + \cos^2 \alpha = \frac{a^2}{a^2 + b^2} + \frac{b^2}{a^2 + b^2} = 1 \text{ And } \tan \alpha = \frac{a}{b} \therefore \alpha = \tan^{-1}\left(\frac{a}{b}\right)$$

$$y = \sin^{-1}(\sin \alpha \cos x - \cos \alpha \sin x)$$

But $\sin(\alpha - x) = \sin \alpha \cos x - \cos \alpha \sin x$

$$y = \sin^{-1}[\sin(\alpha - x)]$$

$$\therefore y = \tan^{-1}\left(\frac{a}{b}\right) - x$$

Differentiate w. r. t. x.

$$\frac{dy}{dx} = \frac{d}{dx} \left[\tan^{-1}\left(\frac{a}{b}\right) - x \right] = 0 - 1 \quad \therefore \frac{dy}{dx} = -1$$

Ex. 5 : Differentiate the following w. r. t. x .

(i) $\sin^{-1}\left(\frac{2x}{1+x^2}\right)$

(ii) $\cos^{-1}(2x\sqrt{1-x^2})$

(iii) $\operatorname{cosec}^{-1}\left(\frac{1}{3x-4x^3}\right)$

(iv) $\tan^{-1}\left(\frac{2e^x}{1-e^{2x}}\right)$

(v) $\cos^{-1}\left(\frac{1-9x^2}{1+9x^2}\right)$

(vi) $\cos^{-1}\left(\frac{2^x-2^{-x}}{2^x+2^{-x}}\right)$

(vii) $\tan^{-1}\left(\sqrt{\frac{3-x}{3+x}}\right)$

(viii) $\sin^{-1}\left(\frac{5\sqrt{1-x^2}-12x}{13}\right)$

(ix) $\sin^{-1}\left(\frac{2^{x+1}}{1+4^x}\right)$

Solution :

(i) Let $y = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$

Put $x = \tan \theta \therefore \theta = \tan^{-1} x$

$\therefore y = \sin^{-1}\left(\frac{2 \tan \theta}{1 + \tan^2 \theta}\right)$

$y = \sin^{-1}(\sin 2\theta) = 2\theta$

$\therefore y = 2 \tan^{-1} x$

Differentiate w. r. t. x .

$$\frac{dy}{dx} = 2 \frac{d}{dx} (\tan^{-1} x)$$

$\therefore \frac{dy}{dx} = \frac{2}{1+x^2}$

(ii) Let $y = \cos^{-1}(2x\sqrt{1-x^2})$

Put $x = \sin \theta \therefore \theta = \sin^{-1} x$

$\therefore y = \cos^{-1}(2 \sin \theta \sqrt{1 - \sin^2 \theta})$

$y = \cos^{-1}(2 \sin \theta \sqrt{\cos^2 \theta})$

$y = \cos^{-1}(2 \sin \theta \cos \theta) = \cos^{-1}(\sin 2\theta)$

$y = \cos^{-1}\left[\cos\left(\frac{\pi}{2} - 2\theta\right)\right] = \frac{\pi}{2} - 2\theta$

$\therefore y = \frac{\pi}{2} - 2 \sin^{-1} x$

Differentiate w. r. t. x .

$$\frac{dy}{dx} = \frac{d}{dx} \left(\frac{\pi}{2} - 2 \sin^{-1} x \right)$$

$$\frac{dy}{dx} = 0 - \frac{2 \times 1}{\sqrt{1-x^2}}$$

$\therefore \frac{dy}{dx} = -\frac{2}{\sqrt{1-x^2}}$

(iii) Let $y = \operatorname{cosec}^{-1}\left(\frac{1}{3x-4x^3}\right)$

$y = \sin^{-1}(3x-4x^3)$

Put $x = \sin \theta \therefore \theta = \sin^{-1} x$

$y = \sin^{-1}(3 \sin \theta - 4 \sin^3 \theta)$

$y = \sin^{-1}(\sin 3\theta) = 3\theta$

$\therefore y = 3 \sin^{-1} x$

Differentiate w. r. t. x .

$$\frac{dy}{dx} = 3 \frac{d}{dx} (\sin^{-1} x)$$

$\therefore \frac{dy}{dx} = \frac{3}{\sqrt{1-x^2}}$

(iv) Let $y = \tan^{-1}\left(\frac{2e^x}{1-e^{2x}}\right)$

Put $e^x = \tan \theta \therefore \theta = \tan^{-1}(e^x)$

$y = \tan^{-1}\left(\frac{2 \tan \theta}{1 + \tan^2 \theta}\right)$

$y = \tan^{-1}(\tan 2\theta) = 2\theta$

$\therefore y = 2 \tan^{-1}(e^x)$

Differentiate w. r. t. x .

$$\frac{dy}{dx} = 2 \frac{d}{dx} [\tan^{-1}(e^x)]$$

$\therefore \frac{dy}{dx} = \frac{2}{1+(e^x)^2} \frac{d}{dx}(e^x) = \frac{2e^x}{1+e^{2x}}$

$$(v) \text{ Let } y = \cos^{-1} \left(\frac{1 - 9x^2}{1 + 9x^2} \right)$$

$$y = \cos^{-1} \left(\frac{1 - (3x)^2}{1 + (3x)^2} \right)$$

$$\text{Put } 3x = \tan \theta \therefore \theta = \tan^{-1}(3x)$$

$$\therefore y = \cos^{-1} \left[\frac{1 - \tan^2 \theta}{1 + \tan^2 \theta} \right]$$

$$y = \cos^{-1}(\cos 2\theta) = 2\theta$$

$$y = 2 \tan^{-1}(3x)$$

Differentiate *w. r. t. x.*

$$\frac{dy}{dx} = 2 \frac{d}{dx} [\tan^{-1}(3x)]$$

$$\frac{dy}{dx} = \frac{2}{1 + (3x)^2} \frac{d}{dx}(3x)$$

$$\therefore \frac{dy}{dx} = \frac{6}{1 + 9x^2}$$

$$(vi) \text{ Let } y = \cos^{-1} \left(\frac{2^x - 2^{-x}}{2^x + 2^{-x}} \right)$$

$$y = \cos^{-1} \left(\frac{2^x(2^x - 2^{-x})}{2^x(2^x + 2^{-x})} \right) \dots \text{ [Multiply \& Divide by } 2^x]$$

$$y = \cos^{-1} \left(\frac{2^{2x} - 1}{2^{2x} + 1} \right) = \cos^{-1} \left[-\frac{1 - (2^x)^2}{1 + (2^x)^2} \right]$$

$$\text{Put } 2^x = \tan \theta \therefore \theta = \tan^{-1}(2^x)$$

$$\therefore y = \cos^{-1} \left[-\frac{1 - \tan^2 \theta}{1 + \tan^2 \theta} \right] = \cos^{-1}[-\cos 2\theta]$$

$$y = \cos^{-1}[\cos(\pi - 2\theta)] = \pi - 2\theta$$

$$y = \pi - 2 \tan^{-1}(2^x)$$

Differentiate *w. r. t. x.*

$$\frac{dy}{dx} = \frac{d}{dx} [\pi - 2 \tan^{-1}(2^x)]$$

$$\frac{dy}{dx} = 0 - \frac{2}{1 + (2^x)^2} \frac{d}{dx}(2^x) = -\frac{2 \cdot 2^x \cdot \log 2}{1 + 2^{2x}}$$

$$\therefore \frac{dy}{dx} = -\frac{2^{x+1} \log 2}{1 + 2^{2x}}$$

$$(vii) \text{ Let } y = \tan^{-1} \left(\sqrt{\frac{3-x}{3+x}} \right)$$

$$\text{Put } x = 3 \cos 2\theta \therefore \theta = \frac{1}{2} \cos^{-1} \left(\frac{x}{3} \right)$$

$$\therefore y = \tan^{-1} \left[\sqrt{\frac{3 - 3 \cos 2\theta}{3 + 3 \cos 2\theta}} \right] = \tan^{-1} \left[\sqrt{\frac{3(1 - \cos 2\theta)}{3(1 + \cos 2\theta)}} \right] = \tan^{-1} \left[\sqrt{\frac{2 \sin^2 \theta}{2 \cos^2 \theta}} \right]$$

$$y = \tan^{-1}(\sqrt{\tan^2 \theta}) = \tan^{-1}(\tan \theta)$$

$$y = \theta = \frac{1}{2} \cos^{-1} \left(\frac{x}{3} \right)$$

Differentiate *w. r. t. x.*

$$\frac{dy}{dx} = \frac{1}{2} \cdot \frac{d}{dx} \left[\cos^{-1} \left(\frac{x}{3} \right) \right]$$

$$= \frac{1}{2} \left[-\frac{1}{\sqrt{1 - \left(\frac{x}{3}\right)^2}} \right] \frac{d}{dx} \left(\frac{x}{3} \right) = -\frac{1}{2} \times \frac{1}{\sqrt{\frac{9-x^2}{9}}} \times \frac{1}{3}$$

$$= -\frac{1}{2} \times \frac{1}{\frac{\sqrt{9-x^2}}{3}} \times \frac{1}{3}$$

$$\therefore \frac{dy}{dx} = -\frac{1}{2\sqrt{9-x^2}}$$

(viii) Let $y = \sin^{-1} \left(\frac{5\sqrt{1-x^2} - 12x}{13} \right)$

Put $x = \sin \theta \therefore \theta = \sin^{-1} x$

$$\therefore y = \sin^{-1} \left(\frac{5\sqrt{1-\sin^2\theta} - 12\sin\theta}{13} \right) = \sin^{-1} \left(\frac{5\sqrt{\cos^2\theta} - 12\sin\theta}{13} \right) = \sin^{-1} \left(\frac{5\cos\theta - 12\sin\theta}{13} \right)$$

$$\therefore y = \sin^{-1} \left(\frac{5}{13} \cos\theta - \frac{12}{13} \sin\theta \right)$$

Put $\frac{5}{13} = \sin \alpha, \frac{12}{13} = \cos \alpha$

Also, $\sin^2 \alpha + \cos^2 \alpha = \frac{25}{169} + \frac{144}{169} = 1$

And $\tan \alpha = \frac{5}{12} \therefore \alpha = \tan^{-1} \left(\frac{5}{12} \right)$

$$y = \sin^{-1} (\sin \alpha \cos \theta - \cos \alpha \sin \theta) = \sin^{-1} [\sin (\alpha - \theta)] = (\alpha - \theta)$$

$$\therefore y = \tan^{-1} \left(\frac{5}{12} \right) - \sin^{-1} x$$

Differentiate *w. r. t. x*.

$$\frac{dy}{dx} = \frac{d}{dx} \left[\tan^{-1} \left(\frac{5}{12} \right) - \sin^{-1} x \right] = 0 - \frac{1}{\sqrt{1-x^2}}$$

$$\therefore \frac{dy}{dx} = -\frac{1}{\sqrt{1-x^2}}$$

(ix) Let $y = \sin^{-1} \left(\frac{2^{x+1}}{1+4^x} \right) = \sin^{-1} \left(\frac{2 \cdot 2^x}{1+(2^x)^2} \right)$

Put $2^x = \tan \theta \therefore \theta = \tan^{-1} (2^x)$

$$\therefore y = \sin^{-1} \left(\frac{2 \tan \theta}{1 + \tan^2 \theta} \right) = \sin^{-1} (\sin 2\theta) = 2\theta = 2 \tan^{-1} (2^x)$$

Differentiate *w. r. t. x*.

$$\frac{dy}{dx} = 2 \frac{d}{dx} [\tan^{-1} (2^x)] = \frac{2}{1+(2^x)^2} \cdot \frac{d}{dx} (2^x) = \frac{2}{1+2^{2x}} (2^x \cdot \log 2)$$

$$\therefore \frac{dy}{dx} = \frac{2^{x+1} \log 2}{1+4^x}$$

Ex. 6 : Differentiate the following *w. r. t. x*.

(i) $\tan^{-1} \left(\frac{4x}{1+21x^2} \right)$

(ii) $\tan^{-1} \left(\frac{7x}{1-12x^2} \right)$

(iii) $\cot^{-1} \left(\frac{b \sin x - a \cos x}{a \sin x + b \cos x} \right)$

(iv) $\tan^{-1} \left(\frac{5x+1}{3-x-6x^2} \right)$

Solution :

$$\begin{aligned} \text{(i) Let } y &= \tan^{-1}\left(\frac{4x}{1+21x^2}\right) \\ &= \tan^{-1}\left(\frac{7x-3x}{1+(7x)(3x)}\right) \\ y &= \tan^{-1}(7x) - \tan^{-1}(3x) \end{aligned}$$

Differentiate w. r. t. x.

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx} [\tan^{-1}(7x) - \tan^{-1}(3x)] \\ &= \frac{d}{dx} [\tan^{-1}(7x)] - \frac{d}{dx} [\tan^{-1}(3x)] \\ &= \frac{1}{1+(7x)^2} \cdot \frac{d}{dx}(7x) - \frac{1}{1+(3x)^2} \cdot \frac{d}{dx}(3x) \\ \therefore \frac{dy}{dx} &= \frac{7}{1+49x^2} - \frac{3}{1+9x^2} \end{aligned}$$

$$\begin{aligned} \text{(ii) Let } y &= \tan^{-1}\left(\frac{7x}{1-12x^2}\right) \\ &= \tan^{-1}\left(\frac{3x+4x}{1-(3x)(4x)}\right) \\ y &= \tan^{-1}(3x) + \tan^{-1}(4x) \end{aligned}$$

Differentiate w. r. t. x.

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx} [\tan^{-1}(3x) + \tan^{-1}(4x)] \\ &= \frac{d}{dx} [\tan^{-1}(3x)] + \frac{d}{dx} [\tan^{-1}(4x)] \\ &= \frac{1}{1+(3x)^2} \cdot \frac{d}{dx}(3x) + \frac{1}{1+(4x)^2} \cdot \frac{d}{dx}(4x) \\ \therefore \frac{dy}{dx} &= \frac{3}{1+9x^2} + \frac{4}{1+16x^2} \end{aligned}$$

$$\begin{aligned} \text{(iii) Let } y &= \cot^{-1}\left(\frac{b \sin x - a \cos x}{a \sin x + b \cos x}\right) = \tan^{-1}\left(\frac{a \sin x + b \cos x}{b \sin x - a \cos x}\right) = \tan^{-1}\left(\frac{\frac{a}{b} + \cot x}{1 - \left(\frac{a}{b}\right)(\cot x)}\right) \\ &= \tan^{-1}\left(\frac{a}{b}\right) + \tan^{-1}(\cot x) = \tan^{-1}\left(\frac{a}{b}\right) + \tan^{-1}\left[\tan\left(\frac{\pi}{2} - x\right)\right] \\ y &= \tan^{-1}\left(\frac{a}{b}\right) + \frac{\pi}{2} - x \end{aligned}$$

Differentiate w. r. t. x.

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx} \left[\tan^{-1}\left(\frac{a}{b}\right) + \frac{\pi}{2} - x \right] \\ &= \frac{d}{dx} \left[\tan^{-1}\left(\frac{a}{b}\right) \right] + \frac{d}{dx} \left(\frac{\pi}{2} \right) - \frac{d}{dx}(x) \\ &= 0 + 0 - 1 \end{aligned}$$

$$\therefore \frac{dy}{dx} = -1$$

$$\begin{aligned} \text{(iv) Let } y &= \tan^{-1}\left(\frac{5x+1}{3-x-6x^2}\right) = \tan^{-1}\left(\frac{5x+1}{1+2-x-6x^2}\right) = \tan^{-1}\left(\frac{5x+1}{1-(6x^2+x-2)}\right) \\ &= \tan^{-1}\left(\frac{5x+1}{1-(6x^2+4x-3x-2)}\right) = \tan^{-1}\left(\frac{5x+1}{1-[2x(3x+2)-(3x+2)]}\right) \\ &= \tan^{-1}\left(\frac{5x+1}{1-(3x+2)(2x-1)}\right) = \tan^{-1}\left(\frac{(3x+2)+(2x-1)}{1-(3x+2)(2x-1)}\right) \end{aligned}$$

$$y = \tan^{-1}(3x + 2) + \tan^{-1}(2x - 1)$$

Differentiate w. r. t. x .

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx} [\tan^{-1}(3x + 2) + \tan^{-1}(2x - 1)] \\ &= \frac{d}{dx} [\tan^{-1}(3x + 2)] + \frac{d}{dx} [\tan^{-1}(2x - 1)] \\ &= \frac{1}{1 + (3x + 2)^2} \cdot \frac{d}{dx} (3x + 2) + \frac{1}{1 + (2x - 1)^2} \cdot \frac{d}{dx} (2x - 1) \\ \therefore \frac{dy}{dx} &= \frac{3}{1 + (3x + 2)^2} + \frac{2}{1 + (2x - 1)^2} \end{aligned}$$

EXERCISE 1.2

- (1) Find the derivative of the function $y = f(x)$ using the derivative of the inverse function $x = f^{-1}(y)$ in the following
- (i) $y = \sqrt{x}$ (ii) $y = \sqrt{2 - \sqrt{x}}$
 (iii) $y = \sqrt[3]{x-2}$ (iv) $y = \log(2x - 1)$
 (v) $y = 2x + 3$ (vi) $y = e^x - 3$
 (vii) $y = e^{2x-3}$ (viii) $y = \log_2\left(\frac{x}{2}\right)$
- (2) Find the derivative of the inverse function of the following
- (i) $y = x^2 \cdot e^x$ (ii) $y = x \cos x$
 (iii) $y = x \cdot 7^x$ (iv) $y = x^2 + \log x$
 (v) $y = x \log x$
- (3) Find the derivative of the inverse of the following functions, and also find their value at the points indicated against them.
- (i) $y = x^5 + 2x^3 + 3x$, at $x = 1$
 (ii) $y = e^x + 3x + 2$, at $x = 0$
 (iii) $y = 3x^2 + 2 \log x^3$, at $x = 1$
 (iv) $y = \sin(x - 2) + x^2$, at $x = 2$
- (4) If $f(x) = x^3 + x - 2$, find $(f^{-1})'(0)$.
- (5) Using derivative prove
- (i) $\tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}$
 (ii) $\sec^{-1} x + \operatorname{cosec}^{-1} x = \frac{\pi}{2} \dots$ [for $|x| \geq 1$]
- (6) Differentiate the following w. r. t. x .
- (i) $\tan^{-1}(\log x)$ (ii) $\operatorname{cosec}^{-1}(e^{-x})$
 (iii) $\cot^{-1}(x^3)$ (iv) $\cot^{-1}(4^x)$
 (v) $\tan^{-1}(\sqrt{x})$ (vi) $\sin^{-1}\left(\sqrt{\frac{1+x^2}{2}}\right)$
 (vii) $\cos^{-1}(1 - x^2)$ (viii) $\sin^{-1}(x^{\frac{3}{2}})$
 (ix) $\cos^3[\cos^{-1}(x^3)]$ (x) $\sin^4[\sin^{-1}(\sqrt{x})]$
- (7) Differentiate the following w. r. t. x .
- (i) $\cot^{-1}[\cot(e^{x^2})]$
 (ii) $\operatorname{cosec}^{-1}\left(\frac{1}{\cos(5^x)}\right)$
 (iii) $\cos^{-1}\left(\sqrt{\frac{1 + \cos x}{2}}\right)$
 (iv) $\cos^{-1}\left(\sqrt{\frac{1 - \cos(x^2)}{2}}\right)$
 (v) $\tan^{-1}\left(\frac{1 - \tan(\frac{x}{2})}{1 + \tan(\frac{x}{2})}\right)$
 (vi) $\operatorname{cosec}^{-1}\left(\frac{1}{4 \cos^3 2x - 3 \cos 2x}\right)$
 (vii) $\tan^{-1}\left(\frac{1 + \cos(\frac{x}{3})}{\sin(\frac{x}{3})}\right)$
 (viii) $\cot^{-1}\left(\frac{\sin 3x}{1 + \cos 3x}\right)$

$$(ix) \tan^{-1} \left(\frac{\cos 7x}{1 + \sin 7x} \right)$$

$$(x) \tan^{-1} \left(\sqrt{\frac{1 + \cos x}{1 - \cos x}} \right)$$

$$(xi) \tan^{-1} (\operatorname{cosec} x + \cot x)$$

$$(xii) \cot^{-1} \left(\frac{\sqrt{1 + \sin \left(\frac{4x}{3}\right)} + \sqrt{1 - \sin \left(\frac{4x}{3}\right)}}{\sqrt{1 + \sin \left(\frac{4x}{3}\right)} - \sqrt{1 - \sin \left(\frac{4x}{3}\right)}} \right)$$

(8) Differentiate the following w. r. t. x .

$$(i) \sin^{-1} \left(\frac{4 \sin x + 5 \cos x}{\sqrt{41}} \right)$$

$$(ii) \cos^{-1} \left(\frac{\sqrt{3} \cos x - \sin x}{2} \right)$$

$$(iii) \sin^{-1} \left(\frac{\cos \sqrt{x} + \sin \sqrt{x}}{\sqrt{2}} \right)$$

$$(iv) \cos^{-1} \left(\frac{3 \cos 3x - 4 \sin 3x}{5} \right)$$

$$(v) \cos^{-1} \left(\frac{3 \cos (e^x) + 2 \sin (e^x)}{\sqrt{13}} \right)$$

$$(vi) \operatorname{cosec}^{-1} \left(\frac{10}{6 \sin (2^x) - 8 \cos (2^x)} \right)$$

(9) Differentiate the following w. r. t. x .

$$(i) \cos^{-1} \left(\frac{1 - x^2}{1 + x^2} \right) \quad (ii) \tan^{-1} \left(\frac{2x}{1 - x^2} \right)$$

$$(iii) \sin^{-1} \left(\frac{1 - x^2}{1 + x^2} \right) \quad (iv) \sin^{-1} (2x \sqrt{1 - x^2})$$

$$(v) \cos^{-1} (3x - 4x^3) \quad (vi) \cos^{-1} \left(\frac{e^x - e^{-x}}{e^x + e^{-x}} \right)$$

$$(vii) \cos^{-1} \left(\frac{1 - 9^x}{1 + 9^x} \right) \quad (viii) \sin^{-1} \left(\frac{4^{x + \frac{1}{2}}}{1 + 2^{4x}} \right)$$

$$(ix) \sin^{-1} \left(\frac{1 - 25x^2}{1 + 25x^2} \right) \quad (x) \sin^{-1} \left(\frac{1 - x^3}{1 + x^3} \right)$$

$$(xi) \tan^{-1} \left(\frac{2x^{\frac{5}{2}}}{1 - x^5} \right) \quad (xii) \cot^{-1} \left(\frac{1 - \sqrt{x}}{1 + \sqrt{x}} \right)$$

(10) Differentiate the following w. r. t. x .

$$(i) \tan^{-1} \left(\frac{8x}{1 - 15x^2} \right) \quad (ii) \cot^{-1} \left(\frac{1 + 35x^2}{2x} \right)$$

$$(iii) \tan^{-1} \left(\frac{2\sqrt{x}}{1 + 3x} \right) \quad (iv) \tan^{-1} \left(\frac{2^{x+2}}{1 - 3(4^x)} \right)$$

$$(v) \tan^{-1} \left(\frac{2^x}{1 + 2^{2x+1}} \right) \quad (vi) \cot^{-1} \left(\frac{a^2 - 6x^2}{5ax} \right)$$

$$(vii) \tan^{-1} \left(\frac{a + b \tan x}{b - a \tan x} \right)$$

$$(viii) \tan^{-1} \left(\frac{5 - x}{6x^2 - 5x - 3} \right)$$

$$(ix) \cot^{-1} \left(\frac{4 - x - 2x^2}{3x + 2} \right)$$

1.3.1 Logarithmic Differentiation

The complicated functions given by formulas that involve products, quotients and powers can often be simplified more quickly by taking the natural logarithms on both the sides. This enables us to use the laws of logarithms to simplify the functions and differentiate easily. Especially when the functions are of the form $y = [f(x)]^{g(x)}$ it is recommended to take logarithms on both the sides which simplifies to $\log y = g(x) \cdot \log [f(x)]$, now it becomes convenient to find the derivative. This process of finding the derivative is called logarithmic differentiation.



SOLVED EXAMPLES

Ex. 1 : Differentiate the following w. r. t. x .

$$(i) \left(\frac{(x^2 + 3)^2 \sqrt[3]{(x^3 + 5)^2}}{\sqrt{(2x^2 + 1)^3}} \right)$$

$$(ii) \frac{e^{x^2} (\tan x)^{\frac{x}{2}}}{(1 + x^2)^{\frac{3}{2}} \cos^3 x}$$

$$(iii) (x+1)^{\frac{3}{2}}(2x+3)^{\frac{5}{2}}(3x+4)^{\frac{2}{3}} \text{ for } x \geq 0 \quad (iv) x^a + x^x + a^x \quad (v) (\sin x)^{\tan x} - x^{\log x}$$

Solution :

$$(i) \text{ Let } y = \left(\frac{(x^2+3)^2 \sqrt[3]{(x^3+5)^2}}{\sqrt{(2x^2+1)^3}} \right)$$

Taking log of both the sides we get,

$$\log y = \log \left(\frac{(x^2+3)^2 \sqrt[3]{(x^3+5)^2}}{\sqrt{(2x^2+1)^3}} \right) = \log \left[\frac{(x^2+3)^2 (x^3+5)^{\frac{2}{3}}}{(2x^2+1)^{\frac{3}{2}}} \right]$$

$$= \log [(x^2+3)^2 (x^3+5)^{\frac{2}{3}}] - \log (2x^2+1)^{\frac{3}{2}}$$

$$= \log (x^2+3)^2 + \log (x^3+5)^{\frac{2}{3}} - \log (2x^2+1)^{\frac{3}{2}}$$

$$\log y = 2 \log (x^2+3) + \frac{2}{3} \log (x^3+5) - \frac{3}{2} \log (2x^2+1)$$

Differentiate w. r. t. x.

$$\frac{d}{dx} (\log y) = \frac{d}{dx} \left[2 \log (x^2+3) + \frac{2}{3} \log (x^3+5) - \frac{3}{2} \log (2x^2+1) \right]$$

$$\frac{1}{y} \frac{dy}{dx} = 2 \cdot \frac{d}{dx} [\log (x^2+3)] + \frac{2}{3} \cdot \frac{d}{dx} [\log (x^3+5)] - \frac{3}{2} \cdot \frac{d}{dx} [\log (2x^2+1)]$$

$$= \frac{2}{x^2+3} \cdot \frac{d}{dx} (x^2+3) + \frac{2}{3(x^3+5)} \cdot \frac{d}{dx} (x^3+5) - \frac{3}{2(2x^2+1)} \cdot \frac{d}{dx} (2x^2+1)$$

$$\frac{dy}{dx} = y \left[\frac{2}{x^2+3} (2x) + \frac{2}{3(x^3+5)} (3x^2) - \frac{3}{2(2x^2+1)} (4x) \right]$$

$$\therefore \frac{dy}{dx} = \frac{(x^2+3)^2 \sqrt[3]{(x^3+5)^2}}{\sqrt{(2x^2+1)^3}} \left[\frac{4x}{x^2+3} + \frac{2x^2}{(x^3+5)} - \frac{6x}{2x^2+1} \right]$$

$$(ii) \text{ Let } y = \frac{e^{x^2} (\tan x)^{\frac{x}{2}}}{(1+x^2)^{\frac{3}{2}} \cos^3 x}$$

Taking log of both the sides we get,

$$\log y = \log \left(\frac{e^{x^2} (\tan x)^{\frac{x}{2}}}{(1+x^2)^{\frac{3}{2}} (\cos x)^3} \right) = \log [e^{x^2} (\tan x)^{\frac{x}{2}}] - \log [(1+x^2)^{\frac{3}{2}} (\cos x)^3]$$

$$= \log e^{x^2} + \log (\tan x)^{\frac{x}{2}} - [\log (1+x^2)^{\frac{3}{2}} + \log (\cos x)^3]$$

$$= x^2 \log e + \frac{x}{2} \log (\tan x) - \frac{3}{2} \log (1+x^2) - 3 \log (\cos x)$$

$$\therefore \log y = x^2 + \frac{x}{2} \log (\tan x) - \frac{3}{2} \log (1+x^2) - 3 \log (\cos x)$$

Differentiate w. r. t. x.

$$\begin{aligned} \frac{d}{dx}(\log y) &= \frac{d}{dx} \left[x^2 + \frac{x}{2} \log(\tan x) - \frac{3}{2} \log(1+x^2) - 3 \log(\cos x) \right] \\ \frac{1}{y} \frac{dy}{dx} &= \frac{d}{dx}(x^2) + \frac{x}{2} \cdot \frac{d}{dx} [\log(\tan x)] + \log(\tan x) \frac{d}{dx} \left(\frac{x}{2} \right) - \frac{3}{2} \frac{d}{dx} [\log(1+x^2)] - 3 \frac{d}{dx} [\log(\cos x)] \\ &= 2x + \frac{x}{2} \cdot \frac{1}{\tan x} \cdot \frac{d}{dx}(\tan x) + \log(\tan x) \frac{1}{2} - \frac{3}{2(1+x^2)} \cdot \frac{d}{dx}(1+x^2) - \frac{3}{\cos x} \cdot \frac{d}{dx}(\cos x) \\ &= 2x + \frac{x}{2} \cdot (\cot x) (\sec^2 x) + \frac{1}{2} \log(\tan x) - \frac{3}{2(1+x^2)} \cdot (2x) - \frac{3}{\cos x} (-\sin x) \\ &= 2x + \frac{x}{2} \times \frac{\cos x}{\sin x} \times \frac{1}{\cos^2 x} + \frac{1}{2} \log(\tan x) - \frac{3x}{1+x^2} + 3 \tan x \\ \frac{dy}{dx} &= y \left[2x + \frac{x}{2 \sin x \cos x} + \frac{1}{2} \log(\tan x) - \frac{3x}{1+x^2} + 3 \tan x \right] \\ \therefore \frac{dy}{dx} &= \frac{e^{x^2} (\tan x)^{\frac{x}{2}}}{(1+x^2)^{\frac{3}{2}} \cos^3 x} \left[2x + x \operatorname{cosec} 2x + \frac{1}{2} \log(\tan x) - \frac{3x}{1+x^2} + 3 \tan x \right] \end{aligned}$$

(iii) Let $y = (x+1)^{\frac{3}{2}} (2x+3)^{\frac{5}{2}} (3x+4)^{\frac{2}{3}}$

Taking log of both the sides we get,

$$\begin{aligned} \log y &= \log \left[(x+1)^{\frac{3}{2}} (2x+3)^{\frac{5}{2}} (3x+4)^{\frac{2}{3}} \right] \\ &= \log (x+1)^{\frac{3}{2}} + \log (2x+3)^{\frac{5}{2}} + \log (3x+4)^{\frac{2}{3}} \\ \log y &= \frac{3}{2} \log (x+1) + \frac{5}{2} \log (2x+3) + \frac{2}{3} \log (3x+4) \end{aligned}$$

Differentiate w. r. t. x.

$$\begin{aligned} \frac{d}{dx}(\log y) &= \frac{d}{dx} \left[\frac{3}{2} \log(x+1) + \frac{5}{2} \log(2x+3) + \frac{2}{3} \log(3x+4) \right] \\ \frac{1}{y} \cdot \frac{dy}{dx} &= \frac{3}{2} \cdot \frac{d}{dx} [\log(x+1)] + \frac{5}{2} \cdot \frac{d}{dx} [\log(2x+3)] + \frac{2}{3} \cdot \frac{d}{dx} [\log(3x+4)] \\ &= \frac{3}{2(x+1)} \cdot \frac{d}{dx}(x+1) + \frac{5}{2(2x+3)} \cdot \frac{d}{dx}(2x+3) + \frac{2}{3(3x+4)} \cdot \frac{d}{dx}(3x+4) \\ \frac{dy}{dx} &= y \left[\frac{3}{2(x+1)} + \frac{5}{2(2x+3)} (2) + \frac{2}{3(3x+4)} (3) \right] \\ \therefore \frac{dy}{dx} &= (x+1)^{\frac{3}{2}} (2x+3)^{\frac{5}{2}} (3x+4)^{\frac{2}{3}} \left[\frac{3}{2(x+1)} + \frac{15}{2(2x+3)} + \frac{2}{3x+4} \right] \end{aligned}$$

(iv) Let $y = x^a + x^x + a^x$

Here the derivatives of x^a and a^x can be found directly but we can not find the derivative of x^x without the use of logarithm. So the given function is split in to two functions, find their derivatives and then add them.



SOLVED EXAMPLES

Ex. 1 : Find $\frac{dy}{dx}$ if

(i) $x^5 + xy^3 + x^2y + y^4 = 4$

(ii) $y^3 + \cos(xy) = x^2 - \sin(x + y)$

(iii) $x^2 + e^{xy} = y^2 + \log(x + y)$

Solution :

(i) Given that : $x^5 + xy^3 + x^2y + y^4 = 4$

Differentiate w. r. t. x .

$$\frac{d}{dx}(x^5) + \frac{d}{dx}(xy^3) + \frac{d}{dx}(x^2y) + \frac{d}{dx}(y^4) = \frac{d}{dx}(4)$$

$$5x^4 + x \frac{d}{dx}(y^3) + y^3 \frac{d}{dx}(x) + x^2 \frac{d}{dx}(y) + y \frac{d}{dx}(x^2) + 4y^3 \frac{d}{dx}(y) = 0$$

$$5x^4 + x(3y^2) \frac{dy}{dx} + y^3(1) + x^2 \frac{dy}{dx} + y(2x) + 4y^3 \frac{dy}{dx} = 0$$

$$x^2 \frac{dy}{dx} + 3xy^2 \frac{dy}{dx} + 4y^3 \frac{dy}{dx} = -5x^4 - 2xy - y^3$$

$$(x^2 + 3xy^2 + 4y^3) \frac{dy}{dx} = -(5x^4 + 2xy + y^3)$$

$$\therefore \frac{dy}{dx} = -\frac{5x^4 + 2xy + y^3}{x^2 + 3xy^2 + 4y^3}$$

(ii) Given that : $y^3 + \cos(xy) = x^2 - \sin(x + y)$

Differentiate w. r. t. x .

$$\frac{d}{dx}(y^3) + \frac{d}{dx}[\cos(xy)] = \frac{d}{dx}(x^2) - \frac{d}{dx}[\sin(x + y)]$$

$$3y^2 \frac{d}{dx}(y) - \sin(xy) \frac{d}{dx}(xy) = 2x - \cos(x + y) \frac{d}{dx}(x + y)$$

$$3y^2 \frac{dy}{dx} - \sin(xy) \left[x \frac{dy}{dx} + y(1) \right] = 2x - \cos(x + y) \left[1 + \frac{dy}{dx} \right]$$

$$3y^2 \frac{dy}{dx} - x \sin(xy) \frac{dy}{dx} - y \sin(xy) = 2x - \cos(x + y) - \cos(x + y) \frac{dy}{dx}$$

$$3y^2 \frac{dy}{dx} - x \sin(xy) \frac{dy}{dx} + \cos(x + y) \frac{dy}{dx} = 2x + y \sin(xy) - \cos(x + y)$$

$$[3y^2 - x \sin(xy) + \cos(x + y)] \frac{dy}{dx} = 2x + y \sin(xy) - \cos(x + y)$$

$$\therefore \frac{dy}{dx} = \frac{2x + y \sin(xy) - \cos(x + y)}{3y^2 - x \sin(xy) + \cos(x + y)}$$

(iii) Given that : $x^2 + e^{xy} = y^2 + \log(x + y)$

Recall that $\frac{d}{dx} g(f(x)) = g'(f(x)) \cdot \frac{d}{dx} f(x)$

Differentiate w. r. t. x .

$$\frac{d}{dx} (x^2) + \frac{d}{dx} [e^{xy}] = \frac{d}{dx} (y^2) + \frac{d}{dx} [\log(x + y)]$$

$$2x + e^{xy} \frac{d}{dx} (xy) = 2y \frac{dy}{dx} + \frac{1}{x + y} \frac{d}{dx} (x + y)$$

$$2x + e^{xy} \left[x \frac{dy}{dx} + y(1) \right] = 2y \frac{dy}{dx} + \frac{1}{x + y} \left[1 + \frac{dy}{dx} \right]$$

$$2x + xe^{xy} \frac{dy}{dx} + ye^{xy} = 2y \frac{dy}{dx} + \frac{1}{x + y} + \frac{1}{x + y} \cdot \frac{dy}{dx}$$

$$2x + ye^{xy} - \frac{1}{x + y} = 2y \frac{dy}{dx} - xe^{xy} \frac{dy}{dx} + \frac{1}{x + y} \cdot \frac{dy}{dx}$$

$$2x + ye^{xy} - \frac{1}{x + y} = \left[2y - xe^{xy} + \frac{1}{x + y} \right] \frac{dy}{dx}$$

$$\frac{2x(x + y) + ye^{xy}(x + y) - 1}{x + y} = \left[\frac{2y(x + y) - xe^{xy}(x + y) + 1}{x + y} \right] \frac{dy}{dx}$$

$$\therefore \frac{dy}{dx} = \frac{2x(x + y) + ye^{xy}(x + y) - 1}{2y(x + y) - xe^{xy}(x + y) + 1}$$

Ex. 2 : Find $x^m \cdot y^n = (x + y)^{m+n}$, then prove that $\frac{dy}{dx} = \frac{y}{x}$.

Solution : Given that : $x^m \cdot y^n = (x + y)^{m+n}$

Taking log on both the sides, we get

$$\log [x^m \cdot y^n] = \log [(x + y)^{m+n}]$$

$$m \log x + n \log y = (m + n) \log (x + y)$$

Differentiate w. r. t. x .

$$m \frac{d}{dx} (\log x) + n \frac{d}{dx} (\log y) = (m + n) \frac{d}{dx} [\log (x + y)]$$

$$\frac{m}{x} + \frac{n}{y} \cdot \frac{dy}{dx} = \frac{m + n}{x + y} \cdot \frac{d}{dx} (x + y)$$

$$\frac{m}{x} + \frac{n}{y} \cdot \frac{dy}{dx} = \frac{m + n}{x + y} \cdot \left[1 + \frac{dy}{dx} \right]$$

$$\frac{m}{x} + \frac{n}{y} \cdot \frac{dy}{dx} = \frac{m + n}{x + y} + \frac{m + n}{x + y} \cdot \frac{dy}{dx}$$

$$\frac{n}{y} \cdot \frac{dy}{dx} - \frac{m + n}{x + y} \cdot \frac{dy}{dx} = \frac{m + n}{x + y} - \frac{m}{x}$$

$$2y \frac{dy}{dx} - \frac{dy}{dx} = \sec^2 x$$

$$(2y - 1) \frac{dy}{dx} = \sec^2 x$$

$$\therefore \frac{dy}{dx} = \frac{\sec^2 x}{2y - 1}$$

Ex. 6 : If $\sqrt{1-x^2} + \sqrt{1-y^2} = a(x-y)$, then show that $\frac{dy}{dx} = \sqrt{\frac{1-y^2}{1-x^2}}$.

Solution : Given that : $\sqrt{1-x^2} + \sqrt{1-y^2} = a(x-y)$ (I)

Put $x = \sin \alpha$, $y = \sin \beta$

$\therefore \alpha = \sin^{-1} x$, $\beta = \sin^{-1} y$

Equation (I) becomes,

$$\sqrt{1-\sin^2 \alpha} + \sqrt{1-\sin^2 \beta} = a(\sin \alpha - \sin \beta)$$

$$\cos \alpha + \cos \beta = a(\sin \alpha - \sin \beta)$$

$$2 \cos \left(\frac{\alpha + \beta}{2} \right) \cos \left(\frac{\alpha - \beta}{2} \right) = 2a \cos \left(\frac{\alpha + \beta}{2} \right) \sin \left(\frac{\alpha - \beta}{2} \right)$$

$$\cos \left(\frac{\alpha - \beta}{2} \right) = a \sin \left(\frac{\alpha + \beta}{2} \right) \Rightarrow \cot \left(\frac{\alpha - \beta}{2} \right) = a$$

$$\frac{\alpha - \beta}{2} = \cot^{-1} a \quad \therefore \alpha - \beta = 2 \cot^{-1} a$$

$$\therefore \sin^{-1} x - \sin^{-1} y = 2 \cot^{-1} a$$

Differentiate w. r. t. x

$$\frac{d}{dx} (\sin^{-1} x) - \frac{d}{dx} (\sin^{-1} y) = \frac{d}{dx} (2 \cot^{-1} a)$$

$$\frac{1}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-y^2}} \cdot \frac{dy}{dx} = 0$$

$$\therefore \frac{dy}{dx} = \sqrt{\frac{1-y^2}{1-x^2}}$$

EXERCISE 1.3

(1) Differentiate the following w. r. t. x

(i) $\frac{(x+1)^2}{(x+2)^3(x+3)^4}$

(ii) $\sqrt[3]{\frac{4x-1}{(2x+3)(5-2x)^2}}$

(iii) $(x^2+3)^{\frac{3}{2}} \cdot \sin^3 2x \cdot 2^{x^2}$

(iv) $\frac{(x^2+2x+2)^{\frac{3}{2}}}{(\sqrt{x}+3)^3(\cos x)^x}$

(v) $\frac{x^5 \cdot \tan^3 4x}{\sin^2 3x}$

(vii) $(\sin x)^x$

(vi) $x^{\tan^{-1} x}$

(viii) $\sin x^x$

(2) Differentiate the following w. r. t. x .

- (i) $x^e + x^x + e^x + e^e$ (ii) $x^{x^x} + e^{x^x}$
 (iii) $(\log x)^x - (\cos x)^{\cot x}$
 (iv) $x^{e^x} + (\log x)^{\sin x}$ (v) $e^{\tan x} + (\log x)^{\tan x}$
 (vi) $(\sin x)^{\tan x} + (\cos x)^{\cot x}$
 (vii) $10^{x^x} + x^{x^{10}} + x^{10^x}$
 (viii) $[(\tan x)^{\tan x}]^{\tan x}$ at $x = \frac{\pi}{4}$

(3) Find $\frac{dy}{dx}$ if

- (i) $\sqrt{x} + \sqrt{y} = \sqrt{a}$ (ii) $x\sqrt{x} + y\sqrt{y} = a\sqrt{a}$
 (iii) $x + \sqrt{xy} + y = 1$
 (iv) $x^3 + x^2y + xy^2 + y^3 = 81$
 (v) $x^2y^2 - \tan^{-1}\sqrt{x^2 + y^2} = \cot^{-1}\sqrt{x^2 + y^2}$
 (vi) $xe^y + ye^x = 1$ (vii) $e^{x+y} = \cos(x-y)$
 (viii) $\cos(xy) = x + y$ (ix) $e^{e^{x-y}} = \frac{x}{y}$
 (x) $x + \sin(x+y) = y - \cos(x-y)$

(4) Show that $\frac{dy}{dx} = \frac{y}{x}$ in the following,

where a and p are constants.

- (i) $x^7y^5 = (x+y)^{12}$
 (ii) $x^p y^4 = (x+y)^{p+4}, p \in \mathbb{N}$
 (iii) $\sec\left(\frac{x^5 + y^5}{x^5 - y^5}\right) = a^2$
 (iv) $\tan^{-1}\left(\frac{3x^2 - 4y^2}{3x^2 + 4y^2}\right) = a^2$
 (v) $\cos^{-1}\left(\frac{7x^4 + 5y^4}{7x^4 - 5y^4}\right) = \tan^{-1} a$
 (vi) $\log\left(\frac{x^{20} - y^{20}}{x^{20} + y^{20}}\right) = 20$
 (vii) $e^{\frac{x^2 - y^2}{x^2 + y^2}} = a$
 (viii) $\sin\left(\frac{x^3 - y^3}{x^3 + y^3}\right) = a^3$

(5) (i) If $\log(x+y) = \log(xy) + p$, where p is constant then prove that $\frac{dy}{dx} = -\frac{y^2}{x^2}$.

(ii) If $\log_{10}\left(\frac{x^3 - y^3}{x^3 + y^3}\right) = 2$,
 show that $\frac{dy}{dx} = -\frac{99x^2}{101y^2}$.

(iii) If $\log_5\left(\frac{x^4 + y^4}{x^4 - y^4}\right) = 2$,
 show that $\frac{dy}{dx} = -\frac{12x^3}{13y^3}$.

(iv) If $e^x + e^y = e^{x+y}$, then
 show that $\frac{dy}{dx} = -e^{y-x}$.

(v) If $\sin^{-1}\left(\frac{x^5 - y^5}{x^5 + y^5}\right) = \frac{\pi}{6}$,
 show that $\frac{dy}{dx} = \frac{x^4}{3y^4}$.

(vi) If $x^y = e^{x-y}$, then
 show that $\frac{dy}{dx} = \frac{\log x}{(1 + \log x)^2}$.

(vii) If $y = \sqrt{\cos x + \sqrt{\cos x + \sqrt{\cos x + \dots \infty}}}$,
 then show that $\frac{dy}{dx} = \frac{\sin x}{1 - 2y}$.

(viii) If $y = \sqrt{\log x + \sqrt{\log x + \sqrt{\log x + \dots \infty}}}$,
 then show that $\frac{dy}{dx} = \frac{1}{x(2y - 1)}$.

(ix) If $y = x^{x^{\dots \infty}}$, then
 show that $\frac{dy}{dx} = \frac{y^2}{x(1 - \log y)}$.

(x) If $e^y = y^x$, then
 show that $\frac{dy}{dx} = \frac{(\log y)^2}{\log y - 1}$.



SOLVED EXAMPLES

Ex. 1 : Find $\frac{dy}{dx}$ if

(i) $x = at^4, y = 2at^2$

(iii) $x = \cos(\log t), y = \log(\cos t)$

(v) $x = \sqrt{1-t^2}, y = \sin^{-1} t$

(ii) $x = t - \sqrt{t}, y = t + \sqrt{t}$

(iv) $x = a(\theta + \sin \theta), y = a(1 - \cos \theta)$

Solution :

(i) Given, $y = 2at^2$

Differentiate w. r. t. t

$$\frac{dy}{dt} = 2a \frac{d}{dt}(t^2) = 2a(2t) = 4at. \dots (I)$$

And, $x = at^4$

Differentiate w. r. t. t

$$\frac{dx}{dt} = a \frac{d}{dt}(t^4) = a(4t^3) = 4at^3. \dots (II)$$

$$\text{Now, } \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{4at}{4at^3} \dots [\text{From (I) and (II)}]$$

$$\therefore \frac{dy}{dx} = \frac{1}{t^2}$$

(ii) Given, $y = t + \sqrt{t}$

Differentiate w. r. t. t

$$\frac{dy}{dt} = \frac{d}{dt}(t + \sqrt{t}) = 1 + \frac{1}{2\sqrt{t}}$$

$$\frac{dy}{dt} = \frac{2\sqrt{t} + 1}{2\sqrt{t}} \dots (I)$$

And, $x = t - \sqrt{t}$

Differentiate w. r. t. t

$$\frac{dx}{dt} = \frac{d}{dt}(t - \sqrt{t}) = 1 - \frac{1}{2\sqrt{t}}$$

$$\frac{dx}{dt} = \frac{2\sqrt{t} - 1}{2\sqrt{t}} \dots (II)$$

$$\text{Now, } \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\frac{2\sqrt{t} + 1}{2\sqrt{t}}}{\frac{2\sqrt{t} - 1}{2\sqrt{t}}} \dots [\text{From (I) and (II)}]$$

$$\therefore \frac{dy}{dx} = \frac{2\sqrt{t} + 1}{2\sqrt{t} - 1}$$

(iii) Given, $y = \log(\cos t)$

Differentiate w. r. t. t

$$\frac{dy}{dt} = \frac{d}{dt}[\log(\cos t)] = \frac{1}{\cos t} \cdot \frac{d}{dt}(\cos t) = \frac{1}{\cos t}(-\sin t) \quad \therefore \frac{dy}{dt} = -\tan t \dots (I)$$

And, $x = \cos(\log t)$

Differentiate w. r. t. t

$$\frac{dx}{dt} = \frac{d}{dt}[\cos(\log t)] = -\sin(\log t) \cdot \frac{d}{dt}(\log t) = -\frac{\sin(\log t)}{t} \quad \therefore \frac{dx}{dt} = -\frac{\sin(\log t)}{t} \dots (II)$$

$$\text{Now, } \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{-\tan t}{-\frac{\sin(\log t)}{t}} \dots [\text{From (I) and (II)}]$$

$$\therefore \frac{dy}{dx} = \frac{t \cdot \tan t}{\sin(\log t)}$$

Ex. 3 : If $x^2 + y^2 = t + \frac{1}{t}$ and $x^4 + y^4 = t^2 + \frac{1}{t^2}$, then show that $x^3y \frac{dy}{dx} = -1$.

Solution :

$$\text{Given that, } x^4 + y^4 = t^2 + \frac{1}{t^2} \quad \dots \text{ (I)}$$

$$\text{And } x^2 + y^2 = t + \frac{1}{t}$$

Squaring both sides,

$$(x^2 + y^2)^2 = \left(t + \frac{1}{t}\right)^2$$

$$x^4 + 2x^2y^2 + y^4 = t^2 + 2 + \frac{1}{t^2}$$

$$x^4 + 2x^2y^2 + y^4 = x^4 + y^4 + 2 \quad \dots \text{ [From (I)]}$$

$$2x^2y^2 = 2 \quad \therefore x^2y^2 = 1 \quad \dots \text{ (II)}$$

Differentiate w. r. t. x

$$\frac{d}{dx}(x^2y^2) = \frac{d}{dx}(1)$$

$$x^2 \frac{d}{dx}(y^2) + y^2 \frac{d}{dx}(x^2) = 0$$

$$x^2(2y) \frac{dy}{dx} + y^2(2x) = 0$$

$$2x^2y \frac{dy}{dx} = -2xy^2 \Rightarrow \frac{dy}{dx} = -\frac{2xy^2}{2x^2y}$$

$$\frac{dy}{dx} = -\frac{x\left(\frac{1}{x^2}\right)}{x^2y} \quad \dots \text{ [From (II)]}$$

$$\therefore \frac{dy}{dx} = \frac{-1}{x^3y} \quad \therefore x^3y \frac{dy}{dx} = -1$$

Ex. 4 : If $x = a\left(t - \frac{1}{t}\right)$ and $y = b\left(t + \frac{1}{t}\right)$, then show that $\frac{dy}{dx} = \frac{b^2x}{a^2y}$.

Solution :

$$\text{Given that, } x = a\left(t - \frac{1}{t}\right) \text{ and } y = b\left(t + \frac{1}{t}\right)$$

$$\text{i.e. } \frac{x}{a} = t - \frac{1}{t} \quad \dots \text{ (I) and } \frac{y}{b} = t + \frac{1}{t} \quad \dots \text{ (II)}$$

Square of (I) – Square of (II) gives,

$$\begin{aligned} \frac{x^2}{a^2} - \frac{y^2}{b^2} &= \left(t - \frac{1}{t}\right)^2 - \left(t + \frac{1}{t}\right)^2 \\ &= t^2 - 2 + \frac{1}{t^2} - t^2 - 2 - \frac{1}{t^2} \end{aligned}$$

$$\therefore \frac{x^2}{a^2} - \frac{y^2}{b^2} = -4$$

Differentiate w. r. t. x

$$\frac{1}{a^2} \cdot \frac{d}{dx}(x^2) - \frac{1}{b^2} \cdot \frac{d}{dx}(y^2) = \frac{d}{dx}(-4)$$

$$\frac{1}{a^2}(2x) - \frac{1}{b^2}(2y) \cdot \frac{dy}{dx} = 0$$

$$\frac{1}{a^2}(2x) - \frac{1}{b^2}(2y) \cdot \frac{dy}{dx} = 0$$

$$\frac{2y}{b^2} \cdot \frac{dy}{dx} = \frac{2x}{a^2} \Rightarrow \frac{dy}{dx} = \frac{b^2x}{a^2y}$$

$$\therefore \frac{dy}{dx} = \frac{b^2x}{a^2y}$$

Ex. 5 : If $x = \sqrt{a^{\sin^{-1}t}}$ and $y = \sqrt{a^{\cos^{-1}t}}$, then show that $\frac{dy}{dx} = -\frac{y}{x}$.

Solution : Given that, $x = \sqrt{a^{\sin^{-1}t}}$ and $y = \sqrt{a^{\cos^{-1}t}}$

$$\text{i.e. } x = \sqrt{a^{\sin^{-1}t}} \quad \dots \text{ (I) and } y = \sqrt{a^{\cos^{-1}t}} \quad \dots \text{ (II)}$$

Differentiate (I) w. r. t. t

$$\frac{dx}{dt} = \frac{d}{dt}\left(\sqrt{a^{\sin^{-1}t}}\right) = \frac{1}{2\sqrt{a^{\sin^{-1}t}}} \cdot \frac{d}{dt}(a^{\sin^{-1}t})$$

$$= \frac{1}{2\sqrt{a^{\sin^{-1}t}}} \cdot a^{\sin^{-1}t} \cdot \log a \frac{d}{dt}(\sin^{-1}t)$$

$$= \frac{a^{\sin^{-1} t} \cdot \log a}{2\sqrt{a^{\sin^{-1} t}} \cdot \sqrt{1-t^2}} \cdot \frac{1}{\sqrt{1-t^2}}$$

$$\frac{dx}{dt} = \frac{\sqrt{a^{\sin^{-1} t}} \cdot \log a}{2\sqrt{1-t^2}} = \frac{x \log a}{2\sqrt{1-t^2}} \dots \text{(III)} \dots \text{[From (I)]}$$

Now $y = \sqrt{a^{\cos^{-1} t}}$

Differentiate (II) w. r. t. t

$$\frac{dy}{dt} = \frac{d}{dt} \left(\sqrt{a^{\cos^{-1} t}} \right) = \frac{1}{2\sqrt{a^{\cos^{-1} t}}} \cdot \frac{d}{dt} (a^{\cos^{-1} t})$$

$$= \frac{1}{2\sqrt{a^{\cos^{-1} t}}} \cdot a^{\cos^{-1} t} \cdot \log a \cdot \frac{d}{dt} (\cos^{-1} t)$$

$$= \frac{a^{\cos^{-1} t} \cdot \log a}{2\sqrt{a^{\cos^{-1} t}}} \left(-\frac{1}{\sqrt{1-t^2}} \right)$$

$$\frac{dy}{dt} = \frac{-\sqrt{a^{\cos^{-1} t}} \cdot \log a}{2\sqrt{1-t^2}} = -\frac{y \log a}{2\sqrt{1-t^2}} \dots \text{(IV)} \dots \text{[From (II)]}$$

Now, $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{-\frac{y \log a}{2\sqrt{1-t^2}}}{\frac{x \log a}{2\sqrt{1-t^2}}} \dots \dots \text{[From (III) and (IV)]}$

$$\therefore \frac{dy}{dx} = -\frac{y}{x}$$

1.4.3 Differentiation of one function with respect to another function :

If y is differentiable function of x , then the derivative of y with respect to x is $\frac{dy}{dx}$.

Similarly, if $u = f(x)$, $v = g(x)$ differentiable function of x , such that $\frac{du}{dx} = f'(x)$ and $\frac{dv}{dx} = g'(x)$

then the derivative of u with respect to v is $\frac{du}{dv} = \frac{\frac{du}{dx}}{\frac{dv}{dx}} = \frac{f'(x)}{g'(x)}$.



SOLVED EXAMPLES

Ex. 1 : Find the derivative of 7^x w. r. t. x^7 .

Solution : Let : $u = 7^x$ and $v = x^7$, then we have to find $\frac{du}{dv}$.

$$\therefore \frac{du}{dv} = \frac{\frac{du}{dx}}{\frac{dv}{dx}} \dots \text{(I)}$$

Now, $u = 7^x$

Differentiate w. r. t. x

$$\frac{du}{dx} = \frac{d}{dx} (7^x) = 7^x \log 7 \dots \text{(II)}$$

And, $v = x^7$

Differentiate w. r. t. x

$$\frac{dv}{dx} = \frac{d}{dx} (x^7) = 7x^6 \dots \text{(III)}$$

Substituting (II) and (III) in (I) we get,

$$\therefore \frac{du}{dv} = \frac{7^x \log 7}{7x^6}$$

Ex. 2 : Find the derivative of $\cos^{-1} x$ w. r. t. $\sqrt{1-x^2}$.

Solution : Let $u = \cos^{-1} x$ and $v = \sqrt{1-x^2}$, then we have to find $\frac{du}{dv}$.

$$\text{i.e. } \frac{du}{dv} = \frac{\frac{du}{dx}}{\frac{dv}{dx}} \quad \dots \text{ (I)}$$

Now, $u = \cos^{-1} x$

Differentiate w. r. t. x

$$\frac{du}{dx} = \frac{d}{dx} (\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}} \quad \dots \text{ (II)}$$

And, $v = \sqrt{1-x^2}$

Differentiate w. r. t. x

$$\frac{dv}{dx} = \frac{d}{dx} (\sqrt{1-x^2}) = \frac{1}{2\sqrt{1-x^2}} \cdot \frac{d}{dx} (1-x^2) = \frac{1}{2\sqrt{1-x^2}} \cdot (-2x)$$

$$\frac{dv}{dx} = -\frac{x}{\sqrt{1-x^2}} \quad \dots \text{ (III)}$$

Substituting (II) and (III) in (I) we get,

$$\frac{du}{dv} = \frac{-\frac{1}{\sqrt{1-x^2}}}{-\frac{x}{\sqrt{1-x^2}}} \quad \therefore \quad \frac{du}{dv} = \frac{1}{x}$$

Ex. 3 : Find the derivative of $\tan^{-1} \left(\frac{\sqrt{1+x^2}-1}{x} \right)$ w. r. t. $\sin^{-1} \left(\frac{2x}{1+x^2} \right)$.

Solution : Let $u = \tan^{-1} \left(\frac{\sqrt{1+x^2}-1}{x} \right)$ and $v = \sin^{-1} \left(\frac{2x}{1+x^2} \right)$, then we have to find $\frac{du}{dv}$.

$$\text{i.e. } \frac{du}{dv} = \frac{\frac{du}{dx}}{\frac{dv}{dx}} \quad \dots \text{ (I)}$$

$$\text{Now, } u = \tan^{-1} \left(\frac{\sqrt{1+x^2}-1}{x} \right)$$

Put $x = \tan \theta \therefore \theta = \tan^{-1} x$

$$u = \tan^{-1} \left(\frac{\sqrt{1+\tan^2 \theta}-1}{\tan \theta} \right) = \tan^{-1} \left(\frac{\sec \theta - 1}{\tan \theta} \right) = \tan^{-1} \left(\frac{\frac{1}{\cos \theta} - 1}{\frac{\sin \theta}{\cos \theta}} \right) = \tan^{-1} \left(\frac{1 - \cos \theta}{\sin \theta} \right)$$

$$= \tan^{-1} \left[\frac{2 \sin^2 \left(\frac{\theta}{2} \right)}{2 \sin \left(\frac{\theta}{2} \right) \cos \left(\frac{\theta}{2} \right)} \right] = \tan^{-1} \left[\tan \left(\frac{\theta}{2} \right) \right]$$

$$u = \frac{\theta}{2} = \frac{1}{2} \tan^{-1} x$$

Differentiate w. r. t. x

$$\frac{du}{dx} = \frac{1}{2} \frac{d}{dx} (\tan^{-1} x) = \frac{1}{2(1+x^2)} \quad \dots \text{ (II)}$$

$$\text{And, } v = \sin^{-1}\left(\frac{2x}{1+x^2}\right) = \sin^{-1}\left(\frac{2 \tan \theta}{1 + \tan^2 \theta}\right) = \sin^{-1}(\sin 2\theta) = 2\theta$$

$$v = 2 \tan^{-1} x$$

Differentiate w. r. t. x

$$\frac{dv}{dx} = 2 \frac{d}{dx}(\tan^{-1} x) = \frac{2}{1+x^2} \quad \dots \text{(III)}$$

Substituting (II) and (III) in (I) we get,

$$\frac{du}{dv} = \frac{\frac{1}{2(1+x^2)}}{\frac{2}{1+x^2}} = \frac{1}{4}$$

EXERCISE 1.4

- (1) Find $\frac{dy}{dx}$ if
- $x = at^2, y = 2at$
 - $x = a \cot \theta, y = b \operatorname{cosec} \theta$
 - $x = \sqrt{a^2 + m^2}, y = \log(a^2 + m^2)$
 - $x = \sin \theta, y = \tan \theta$
 - $x = a(1 - \cos \theta), y = b(\theta - \sin \theta)$
 - $x = \left(t + \frac{1}{t}\right)^a, y = a^t + \frac{1}{t}$,
where $a > 0, a \neq 1$ and $t \neq 0$.
 - $x = \cos^{-1}\left(\frac{2t}{1+t^2}\right), y = \sec^{-1}(\sqrt{1+t^2})$
 - $x = \cos^{-1}(4t^3 - 3t), y = \tan^{-1}\left(\frac{\sqrt{1-t^2}}{t}\right)$
- (2) Find $\frac{dy}{dx}$ if
- $x = \operatorname{cosec}^2 \theta, y = \cot^3 \theta$, at $\theta = \frac{\pi}{6}$
 - $x = a \cos^3 \theta, y = a \sin^3 \theta$, at $\theta = \frac{\pi}{3}$
 - $x = t^2 + t + 1, y = \sin\left(\frac{\pi t}{2}\right) + \cos\left(\frac{\pi t}{2}\right)$,
at $t = 1$
 - $x = 2 \cos t + \cos 2t, y = 2 \sin t - \sin 2t$,
at $t = \frac{\pi}{4}$
 - $x = t + 2 \sin(\pi t), y = 3t - \cos(\pi t)$,
at $t = \frac{1}{2}$
- (3) (i) If $x = a\sqrt{\sec \theta - \tan \theta}, y = a\sqrt{\sec \theta + \tan \theta}$,
then show that $\frac{dy}{dx} = -\frac{y}{x}$.
- (ii) If $x = e^{\sin 3t}, y = e^{\cos 3t}$, then
show that $\frac{dy}{dx} = -\frac{y \log x}{x \log y}$.
- (iii) If $x = \frac{t+1}{t-1}, y = \frac{t-1}{t+1}$, then
show that $y^2 + \frac{dy}{dx} = 0$.
- (iv) If $x = a \cos^3 t, y = a \sin^3 t$, then
show that $\frac{dy}{dx} = -\left(\frac{y}{x}\right)^{\frac{1}{3}}$.
- (v) If $x = 2 \cos^4(t+3), y = 3 \sin^4(t+3)$,
show that $\frac{dy}{dx} = -\sqrt{\frac{3y}{2x}}$.
- (vi) If $x = \log(1+t^2), y = t - \tan^{-1} t$,
show that $\frac{dy}{dx} = \frac{\sqrt{e^x - 1}}{2}$.
- (vii) If $x = \sin^{-1}(e^t), y = \sqrt{1 - e^{2t}}$,
show that $\sin x + \frac{dy}{dx} = 0$.
- (viii) If $x = \frac{2bt}{1+t^2}, y = a\left(\frac{1-t^2}{1+t^2}\right)$,
show that $\frac{dx}{dy} = -\frac{b^2 y}{a^2 x}$.

(4) (i) Differentiate $x \sin x$ w. r. t. $\tan x$.

(ii) Differentiate $\sin^{-1} \left(\frac{2x}{1+x^2} \right)$

w. r. t. $\cos^{-1} \left(\frac{1-x^2}{1+x^2} \right)$.

(iii) Differentiate $\tan^{-1} \left(\frac{x}{\sqrt{1-x^2}} \right)$

w. r. t. $\sec^{-1} \left(\frac{1}{2x^2-1} \right)$.

(iv) Differentiate $\cos^{-1} \left(\frac{1-x^2}{1+x^2} \right)$ w. r. t. $\tan^{-1} x$.

(v) Differentiate 3^x w. r. t. $\log_x 3$.

(vi) Differentiate $\tan^{-1} \left(\frac{\cos x}{1+\sin x} \right)$

w. r. t. $\sec^{-1} x$.

(vii) Differentiate x^x w. r. t. $x^{\sin x}$.

(viii) Differentiate $\tan^{-1} \left(\frac{\sqrt{1+x^2}-1}{x} \right)$

w. r. t. $\tan^{-1} \left(\frac{2x\sqrt{1-x^2}}{1-2x^2} \right)$.

1.5.1 Higher order derivatives :

If $f(x)$ is differentiable function of x on an open interval I , then its derivative $f'(x)$ is also a function on I , so $f'(x)$ may have a derivative of its own, denoted as $(f'(x))' = f''(x)$. This new function $f''(x)$ is called the **second derivative** of $f(x)$. By Leibniz notation, we write the second derivative of

$$y = f(x) \text{ as } y'' = f''(x) = \frac{d}{dx} \left(\frac{dy}{dx} \right) = \frac{d^2y}{dx^2}$$

By method of first principle

$$f'(x) = \lim_{h \rightarrow 0} \left(\frac{f(x+h) - f(x)}{h} \right) = \frac{dy}{dx} \text{ and}$$

$$f''(x) = \lim_{h \rightarrow 0} \left(\frac{f'(x+h) - f'(x)}{h} \right) = \frac{d^2y}{dx^2}$$

Further if $f''(x)$ is a differentiable function of x then its derivative is denoted as $\frac{d}{dx}[f''(x)] = f'''(x)$.

Now the new function $f'''(x)$ is called the **third derivative** of $f(x)$. We write the third of $y = f(x)$ as

$$y''' = f'''(x) = \frac{d}{dx} \left(\frac{d^2y}{dx^2} \right) = \frac{d^3y}{dx^3}. \text{ The } \mathbf{fourth \text{ derivative}}, \text{ is usually denoted by } f^{(4)}(x). \text{ Therefore}$$

$$f^{(4)}(x) = \frac{d^4y}{dx^4}.$$

In general, the n^{th} derivative of $f(x)$, is denoted by $f^{(n)}(x)$ and it obtained by differentiating $f(x)$, n times. So, we can write the n^{th} derivative of $y = f(x)$ as $y^{(n)} = f^{(n)}(x) = \frac{d^ny}{dx^n}$. These are called higher order derivatives.

Note : The higher order derivatives may also be denoted by y_2, y_3, \dots, y_n .

For example: Consider $f(x) = x^3 - x$

Differentiate w. r. t. x

$$f'(x) = \frac{d}{dx} [f(x)] = 3x^2 - 1$$

Differentiate w. r. t. x

$$f''(x) = \frac{d}{dx} [f'(x)] = 6x$$

Differentiate w. r. t. x

$$f'''(x) = \frac{d}{dx} [f''(x)] = 6$$

Which is the slope of the line represented by $f''(x)$. Hence forth all its next derivatives are zero.

Note : From the above example we can deduce one important result that, if $f(x)$ is a polynomial of degree n , then its n^{th} order derivative is a constant and all the onward derivatives are zeros.



SOLVED EXAMPLES

Ex. 1 : Find the second order derivative of the following :

(i) $x^3 + 7x^2 - 2x - 9$

(ii) $x^2 e^x$

(iii) $e^{2x} \sin 3x$

(iv) $x^2 \log x$

(v) $\sin(\log x)$

Solution :

(i) Let $y = x^3 + 7x^2 - 2x - 9$

Differentiate w. r. t. x

$$\frac{dy}{dx} = \frac{d}{dx} (x^3 + 7x^2 - 2x - 9)$$

$$\frac{dy}{dx} = 3x^2 + 14x - 2$$

Differentiate w. r. t. x

$$\frac{d}{dx} \left(\frac{dy}{dx} \right) = \frac{d}{dx} (3x^2 + 14x - 2)$$

$$\frac{d^2y}{dx^2} = 6x + 14$$

(ii) Let $y = x^2 e^x$

Differentiate w. r. t. x

$$\frac{dy}{dx} = \frac{d}{dx} (x^2 e^x)$$

$$\frac{dy}{dx} = x^2 \frac{d}{dx} (e^x) + e^x \frac{d}{dx} (x^2)$$

$$\frac{dy}{dx} = x^2 e^x + 2x e^x = e^x (x^2 + 2x)$$

Differentiate w. r. t. x

$$\frac{d}{dx} \left(\frac{dy}{dx} \right) = \frac{d}{dx} [e^x (x^2 + 2x)]$$

$$\frac{d^2y}{dx^2} = e^x \frac{d}{dx} (x^2 + 2x) + (x^2 + 2x) \frac{d}{dx} (e^x)$$

$$= e^x (2x + 2) + (x^2 + 2x) (e^x)$$

$$= (x^2 + 4x + 2) e^x$$

$$\frac{d^2y}{dx^2} = (x^2 + 4x + 2) e^x$$

(iii) Let $y = e^{2x} \sin 3x$

Differentiate w. r. t. x

$$\frac{dy}{dx} = \frac{d}{dx} (e^{2x} \sin 3x) = e^{2x} \frac{d}{dx} (\sin 3x) + \sin 3x \frac{d}{dx} (e^{2x})$$

$$\frac{dy}{dx} = e^{2x} (\cos 3x) (3) + \sin 3x (e^{2x}) (2)$$

$$\frac{dy}{dx} = e^{2x} (3 \cos 3x + 2 \sin 3x)$$

Differentiate w. r. t. x

$$\frac{d}{dx} \left(\frac{dy}{dx} \right) = \frac{d}{dx} [e^{2x} (3 \cos 3x + 2 \sin 3x)]$$

$$\frac{d^2y}{dx^2} = e^{2x} \frac{d}{dx} (3 \cos 3x + 2 \sin 3x) + (3 \cos 3x + 2 \sin 3x) \frac{d}{dx} (e^{2x})$$

$$= e^{2x} [3 (-\sin 3x) (3) + 2 (\cos 3x)(3)] + (3 \cos 3x + 2 \sin 3x) e^{2x} (2)$$

$$= e^{2x} [-9 \sin 3x + 6 \cos 3x + 6 \cos 3x + 4 \sin 3x]$$

$$\frac{d^2y}{dx^2} = e^{2x} [12 \cos 3x - 5 \sin 3x]$$

(iv) Let $y = x^2 \log x$

Differentiate w. r. t. x

$$\frac{dy}{dx} = \frac{d}{dx} (x^2 \log x)$$

$$\frac{dy}{dx} = x^2 \frac{d}{dx} (\log x) + \log x \frac{d}{dx} (x^2)$$

$$\frac{dy}{dx} = x^2 \cdot \frac{1}{x} + \log x (2x)$$

$$\frac{dy}{dx} = x(1 + 2 \log x)$$

Differentiate w. r. t. x

$$\frac{d}{dx} \left(\frac{dy}{dx} \right) = \frac{d}{dx} [x(1 + 2 \log x)]$$

$$\frac{d^2y}{dx^2} = x \frac{d}{dx} (1 + 2 \log x) + (1 + 2 \log x) \frac{d}{dx} (x)$$

$$= x \cdot \frac{2}{x} + (1 + 2 \log x) (1)$$

$$\frac{d^2y}{dx^2} = 3 + 2 \log x$$

(v) Let $y = \sin (\log x)$

Differentiate w. r. t. x

$$\frac{dy}{dx} = \frac{d}{dx} [\sin (\log x)]$$

$$\frac{dy}{dx} = \cos (\log x) \frac{d}{dx} (\log x)$$

$$\frac{dy}{dx} = \frac{\cos (\log x)}{x}$$

Differentiate w. r. t. x

$$\frac{d}{dx} \left(\frac{dy}{dx} \right) = \frac{d}{dx} \left[\frac{\cos (\log x)}{x} \right]$$

$$\frac{d^2y}{dx^2} = \frac{x \frac{d}{dx} [\cos (\log x)] - \cos (\log x) \frac{d}{dx} (x)}{x^2}$$

$$= \frac{x [-\sin (\log x)] \frac{d}{dx} (\log x) - \cos (\log x) (1)}{x^2}$$

$$= \frac{-\frac{x \sin (\log x)}{x} - \cos (\log x)}{x^2}$$

$$\frac{d^2y}{dx^2} = -\frac{\sin (\log x) + \cos (\log x)}{x^2}$$

(ii) $x = a \cos^3 \theta, y = b \sin^3 \theta$ at $\theta = \frac{\pi}{4}$

Solution :

Given that : $x = a \cos^3 \theta$

Differentiate w. r. t. θ

$$\frac{dx}{d\theta} = \frac{d}{d\theta} (a \cos^3 \theta) = a (3) (\cos^2 \theta) \frac{d}{d\theta} (\cos \theta)$$

$$\frac{dx}{d\theta} = -3a \cos^2 \theta \sin \theta \quad \dots \text{(I)}$$

$y = b \sin^3 \theta$

Differentiate w. r. t. θ

$$\frac{dy}{d\theta} = \frac{d}{d\theta} (b \sin^3 \theta) = b (3) (\sin^2 \theta) \frac{d}{d\theta} (\sin \theta)$$

$$\frac{dy}{d\theta} = 3b \sin^2 \theta \cos \theta \quad \dots \text{(II)}$$

We know that,

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{3b \sin^2 \theta \cos \theta}{-3a \cos^2 \theta \sin \theta} \quad \dots \text{[From (I) and (II)]}$$

$$\therefore \frac{dy}{dx} = -\frac{b}{a} \cdot \tan \theta$$

Differentiate w. r. t. x

$$\frac{d}{dx} \left(\frac{dy}{dx} \right) = -\frac{b}{a} \cdot \frac{d}{dx} (\tan \theta)$$

$$\frac{d^2y}{dx^2} = -\frac{b}{a} \cdot \frac{d}{d\theta} (\tan \theta) \times \frac{d\theta}{dx}$$

$$= -\frac{b}{a} (\sec^2 \theta) \times \frac{1}{\frac{dx}{d\theta}}$$

$$= -\frac{b}{a} (\sec^2 \theta) \times \frac{1}{-3a \cos^2 \theta \sin \theta} \quad \dots \text{[From (I)]}$$

$$\frac{d^2y}{dx^2} = \frac{b}{3a^2} \times \frac{\sec^2 \theta}{\cos^2 \theta \sin \theta}$$

$$\frac{d^2y}{dx^2} = \frac{b \sec^4 \theta}{3a^2 \sin \theta}$$

When $\theta = \frac{\pi}{4}$

$$\left(\frac{d^2y}{dx^2} \right)_{\theta = \frac{\pi}{4}} = \frac{b \sec^4 \left(\frac{\pi}{4} \right)}{3a^2 \sin \left(\frac{\pi}{4} \right)} = \frac{b (\sqrt{2})^4}{3a^2 \left(\frac{1}{\sqrt{2}} \right)}$$

$$\left(\frac{d^2y}{dx^2} \right)_{\theta = \frac{\pi}{4}} = \frac{4 \sqrt{2} b}{3a^2}$$

Ex. 3 : If $ax^2 + 2hxy + by^2 = 0$ then show that $\frac{d^2y}{dx^2} = 0$.

Solution : Given that $ax^2 + 2hxy + by^2 = 0 \quad \dots$ (I)

$$ax^2 + hxy + hxy + by^2 = 0$$

$$x(ax + hy) + y(hx + by) = 0$$

$$y(hx + by) = -x(ax + hy)$$

$$\frac{y}{x} = -\frac{ax + hy}{hx + by} \quad \dots$$
 (II)

Differentiate (I) w. r. t. x

$$a \frac{d}{dx} (x^2) + 2h \frac{d}{dx} (xy) + b \frac{d}{dx} (y^2) = 0$$

$$a(2x) + 2h \left[a \frac{dy}{dx} + y(1) \right] + b(2y) \frac{dy}{dx} = 0$$

$$2 \left[ax + hx \frac{dy}{dx} + hy + by \frac{dy}{dx} \right] = 0$$

$$(hx + by) \frac{dy}{dx} = -ax - hy$$

$$\frac{dy}{dx} = -\frac{ax + hy}{hx + by}$$

From (II), we get

$$\therefore \frac{dy}{dx} = \frac{y}{x} \quad \dots$$
 (III)

Differentiate (III), w. r. t. x

$$\frac{d}{dx} \left(\frac{dy}{dx} \right) = \frac{d}{dx} \left(\frac{y}{x} \right)$$

$$\frac{d^2y}{dx^2} = \frac{x \frac{dy}{dx} - y(1)}{x^2} = \frac{x \left(\frac{y}{x} \right) - y}{x^2} \dots$$
 [From (II)]

$$\therefore \frac{d^2y}{dx^2} = \frac{y - y}{x^2} = 0$$

Ex. 4 : If $y = \cos (m \cos^{-1} x)$ then show that $(1 - x^2) \frac{d^2y}{dx^2} - x \frac{dy}{dx} + m^2y = 0$.

Solution : Given that $y = \cos (m \cos^{-1} x)$

$$\therefore \cos^{-1} y = m \cos^{-1} x$$

Differentiate (I) w. r. t. x

$$\frac{d}{dx} (\cos^{-1} y) = m \frac{d}{dx} (\cos^{-1} x)$$

$$-\frac{1}{\sqrt{1-y^2}} \cdot \frac{dy}{dx} = -\frac{m}{\sqrt{1-x^2}}$$

$$\sqrt{1-x^2} \cdot \frac{dy}{dx} = m \sqrt{1-y^2}$$

Squaring both sides

$$(1-x^2) \cdot \left(\frac{dy}{dx}\right)^2 = m^2 (1-y^2)$$

Differentiate w. r. t. x

$$(1-x^2) \frac{d}{dx} \left(\frac{dy}{dx}\right)^2 + \left(\frac{dy}{dx}\right)^2 \frac{d}{dx} (1-x^2) = m^2 \frac{d}{dx} (1-y^2)$$

$$(1-x^2) \cdot 2 \left(\frac{dy}{dx}\right) \cdot \frac{d}{dx} \left(\frac{dy}{dx}\right) + \left(\frac{dy}{dx}\right)^2 (-2x) = m^2 (-2y) \frac{dy}{dx}$$

$$2(1-x^2) \cdot \frac{dy}{dx} \cdot \frac{d^2y}{dx^2} - 2x \left(\frac{dy}{dx}\right)^2 = -2m^2y \frac{dy}{dx}$$

Dividing throughout by $2 \frac{dy}{dx}$ we get,

$$(1-x^2) \cdot \frac{d^2y}{dx^2} - x \frac{dy}{dx} = -m^2y$$

$$\therefore (1-x^2) \cdot \frac{d^2y}{dx^2} - x \frac{dy}{dx} + m^2y = 0$$

Ex. 5 : If $x = \sin t$, $y = e^{mt}$ then show that $(1-x^2) \frac{d^2y}{dx^2} - x \frac{dy}{dx} - m^2y = 0$.

Solution : Given that $x = \sin t \quad \therefore \quad t = \sin^{-1} x$
and $y = e^{mt} \quad \therefore \quad y = e^{m \sin^{-1} x} \quad \dots$ (I)

Differentiate w. r. t. x

$$\frac{dy}{dx} = \frac{d}{dx} (e^{m \sin^{-1} x}) = e^{m \sin^{-1} x} \cdot m \frac{d}{dx} (\sin^{-1} x)$$

$$\frac{dy}{dx} = \frac{m \cdot e^{m \sin^{-1} x}}{\sqrt{1-x^2}}$$

$$\sqrt{1-x^2} \frac{dy}{dx} = my \quad \dots \text{ [From (I)]}$$

Squaring both sides

$$(1-x^2) \cdot \left(\frac{dy}{dx}\right)^2 = m^2 y^2$$

Differentiate w. r. t. x

$$(1-x^2) \frac{d}{dx} \left(\frac{dy}{dx}\right)^2 + \left(\frac{dy}{dx}\right)^2 \frac{d}{dx} (1-x^2) = m^2 \frac{d}{dx} (y^2)$$

$$(1-x^2) \cdot 2 \left(\frac{dy}{dx}\right) \cdot \frac{d}{dx} \left(\frac{dy}{dx}\right) + \left(\frac{dy}{dx}\right)^2 (-2x) = m^2 (2y) \frac{dy}{dx}$$

$$2(1-x^2) \cdot \frac{dy}{dx} \cdot \frac{d^2y}{dx^2} - 2x \left(\frac{dy}{dx} \right)^2 = 2m^2y \frac{dy}{dx}$$

Dividing throughout by $2 \frac{dy}{dx}$ we get,

$$(1-x^2) \cdot \frac{d^2y}{dx^2} - x \frac{dy}{dx} = m^2y$$

$$\therefore (1-x^2) \cdot \frac{d^2y}{dx^2} - x \frac{dy}{dx} - m^2y = 0$$

1.5.2 Successive differentiation (or n^{th} order derivative) of some standard functions :

Successive Differentiation is the process of differentiating a given function successively for n times and the results of such differentiation are called successive derivatives. The higher order derivatives are of utmost importance in scientific and engineering applications.

There is no general formula to find n^{th} derivative of a function. Because each and every function has it's own specific general formula for it's n^{th} derivative. But there are algorithms to find it.

So, here is the algorithm, for some standard functions.

Let us use the method of mathematical induction wherever applicable.

Step 1 :- Use simple differentiation to get 1st, 2nd and 3rd order derivatives.

Step 2 :- Observe the changes in the coefficients, the angles, the power of the function and the signs of each term etc.

Step 3 :- Express the n^{th} derivative with the help of the patterns of changes that you have observed. This will be your general formula for the n^{th} derivative of the given standard function.



SOLVED EXAMPLES

Ex. 1 : Find the n^{th} derivative of the following :

(i) x^m

(ii) $\frac{1}{ax+b}$

(iii) $\log x$

(iv) $\sin x$

(v) $\cos(ax+b)$

(vi) $e^{ax} \sin(bx+c)$

Solution :

(i) Let $y = x^m$

Differentiate w. r. t. x

$$\frac{dy}{dx} = \frac{d}{dx} (x^m) = mx^{m-1}$$

Differentiate w. r. t. x

$$\frac{d}{dx} \left(\frac{dy}{dx} \right) = m \frac{d}{dx} x^{m-1}$$

$$\frac{d^2y}{dx^2} = m \cdot (m-1) x^{m-2}$$

Differentiate w. r. t. x

$$\frac{d}{dx} \left(\frac{d^2y}{dx^2} \right) = m \cdot (m-1) \frac{d}{dx} (x^{m-2})$$

$$\frac{d^3y}{dx^3} = m \cdot (m-1) \cdot (m-2) x^{m-3}$$

In general n^{th} order derivative will be

$$\frac{d^n y}{dx^n} = m \cdot (m-1) \cdot (m-2) \dots [m - (n-1)] x^{m-n}$$

$$\frac{d^n y}{dx^n} = m \cdot (m-1) \cdot (m-2) \dots [m - n + 1] x^{m-n}$$

case (i) :- If $m > 0$ and $m > n$, then

$$\frac{d^n y}{dx^n} = \frac{m \cdot (m-1) \cdot (m-2) \dots [m - (n-1)] \cdot (m-n) \dots 2 \cdot 1}{(m-n) \cdot [m-n-1] \dots 2 \cdot 1} x^{m-n}$$

$$\frac{d^n y}{dx^n} = \frac{m! \cdot x^{m-n}}{(m-n)!}$$

case (ii) :- If $m > 0$ and $m = n$, then

$$\frac{d^n y}{dx^n} = \frac{n! \cdot x^{m-n}}{(n-n)!} = \frac{n! \cdot x^0}{0!} = n!$$

case (iii) :- If $m > 0$ and $m < n$, then

$$\frac{d^n y}{dx^n} = 0$$

(ii) Let $y = \frac{1}{ax+b}$

Differentiate w. r. t. x

$$\frac{dy}{dx} = \frac{d}{dx} \left(\frac{1}{ax+b} \right) = \frac{-1}{(ax+b)^2} \cdot \frac{d}{dx} (ax+b)$$

$$\frac{dy}{dx} = \frac{(-1) \cdot a}{(ax+b)^2}$$

Differentiate w. r. t. x

$$\frac{d}{dx} \left(\frac{dy}{dx} \right) = (-1)(a) \frac{d}{dx} \left(\frac{1}{(ax+b)^2} \right)$$

$$\frac{d^2 y}{dx^2} = (-1)(a) \frac{-2}{(ax+b)^3} \cdot \frac{d}{dx} (ax+b)$$

$$\frac{d^2 y}{dx^2} = \frac{(-1)^2 \cdot 2 \cdot 1 \cdot a^2}{(ax+b)^3}$$

Differentiate w. r. t. x

$$\frac{d}{dx} \left(\frac{d^2 y}{dx^2} \right) = (-1)^2 \cdot 2 \cdot 1 \cdot a^2 \cdot \frac{d}{dx} \left(\frac{1}{(ax+b)^3} \right)$$

$$\frac{d^3 y}{dx^3} = (-1)^2 \cdot 2 \cdot 1 \cdot a^2 \cdot \frac{-3}{(ax+b)^4} \cdot \frac{d}{dx} (ax+b)$$

$$\frac{d^3 y}{dx^3} = \frac{(-1)^3 \cdot 3 \cdot 2 \cdot 1 \cdot a^3}{(ax+b)^4}$$

In general n^{th} order derivative will be

$$\frac{d^n y}{dx^n} = \frac{(-1)^n \cdot n \cdot (n-1) \dots 2 \cdot 1 \cdot a^n}{(ax+b)^{n+1}}$$

$$\frac{d^n y}{dx^n} = \frac{(-1)^n \cdot n! \cdot a^n}{(ax+b)^{n+1}}$$

(iii) Let $y = \log x$

Differentiate w. r. t. x

$$\frac{dy}{dx} = \frac{d}{dx} (\log x) = \frac{1}{x}$$

Differentiate w. r. t. x

$$\frac{d}{dx} \left(\frac{dy}{dx} \right) = \frac{d}{dx} \left(\frac{1}{x} \right)$$

$$\frac{d^2 y}{dx^2} = \frac{-1}{x^2} = \frac{(-1)^1}{x^2}$$

Differentiate w. r. t. x

$$\frac{d}{dx} \left(\frac{d^2 y}{dx^2} \right) = (-1)^1 \frac{d}{dx} \left(\frac{1}{x^2} \right)$$

$$\frac{d^3 y}{dx^3} = (-1)^1 \left(\frac{-2}{x^3} \right) = \frac{(-1)^2 \cdot 1 \cdot 2}{x^3}$$

In general n^{th} order derivative will be

$$\frac{d^n y}{dx^n} = \frac{(-1)^{n-1} \cdot 1 \cdot 2 \cdot 3 \dots (n-1)}{x^n}$$

$$\frac{d^n y}{dx^n} = \frac{(-1)^{n-1} \cdot (n-1)!}{x^n}$$

(iv) Let $y = \sin x$

Differentiate w. r. t. x

$$\frac{dy}{dx} = \frac{d}{dx} (\sin x) = \cos x$$

$$\frac{dy}{dx} = \sin \left(\frac{\pi}{2} + x \right)$$

Differentiate w. r. t. x

$$\frac{d}{dx} \left(\frac{dy}{dx} \right) = \frac{d}{dx} \left[\sin \left(\frac{\pi}{2} + x \right) \right]$$

$$\frac{d^2y}{dx^2} = \cos \left(\frac{\pi}{2} + x \right) \frac{d}{dx} \left(\frac{\pi}{2} + x \right)$$

$$\frac{d^2y}{dx^2} = \sin \left(\frac{\pi}{2} + \frac{\pi}{2} + x \right) \quad (1)$$

$$\frac{d^2y}{dx^2} = \sin \left(\frac{2\pi}{2} + x \right)$$

Differentiate w. r. t. x

$$\frac{d}{dx} \left(\frac{d^2y}{dx^2} \right) = \frac{d}{dx} \left[\sin \left(\frac{2\pi}{2} + x \right) \right]$$

$$\begin{aligned} \frac{d^3y}{dx^3} &= \cos \left(\frac{2\pi}{2} + x \right) \frac{d}{dx} \left(\frac{2\pi}{2} + x \right) \\ &= \sin \left(\frac{\pi}{2} + \frac{2\pi}{2} + x \right) \quad (1) \end{aligned}$$

$$\frac{d^3y}{dx^3} = \sin \left(\frac{3\pi}{2} + x \right)$$

In general n^{th} order derivative will be

$$\frac{d^n y}{dx^n} = \sin \left(\frac{n\pi}{2} + x \right)$$

(v) Let $y = \cos (ax + b)$

Differentiate w. r. t. x

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx} [\cos (ax + b)] \\ &= -\sin (ax + b) \frac{d}{dx} (ax + b) \end{aligned}$$

$$= \cos \left(\frac{\pi}{2} + ax + b \right) (a)$$

$$\frac{dy}{dx} = a \cos \left(\frac{\pi}{2} + ax + b \right)$$

Differentiate w. r. t. x

$$\frac{d}{dx} \left(\frac{dy}{dx} \right) = \frac{d}{dx} \left[a \cos \left(\frac{\pi}{2} + ax + b \right) \right]$$

$$\frac{d}{dx} \left(\frac{dy}{dx} \right) = a \frac{d}{dx} \left[\cos \left(\frac{\pi}{2} + ax + b \right) \right]$$

$$\begin{aligned} \frac{d^2y}{dx^2} &= a \left[-\sin \left(\frac{\pi}{2} + ax + b \right) \right] \frac{d}{dx} \left(\frac{\pi}{2} + ax + b \right) \\ &= a \cos \left(\frac{\pi}{2} + \frac{\pi}{2} + ax + b \right) (a) \end{aligned}$$

$$\frac{d^2y}{dx^2} = a^2 \cos \left(\frac{2\pi}{2} + ax + b \right)$$

Differentiate w. r. t. x

$$\frac{d}{dx} \left(\frac{d^2y}{dx^2} \right) = \frac{d}{dx} \left[a^2 \cos \left(\frac{2\pi}{2} + ax + b \right) \right]$$

$$\frac{d}{dx} \left(\frac{d^2y}{dx^2} \right) = a^2 \frac{d}{dx} \left[\cos \left(\frac{2\pi}{2} + ax + b \right) \right]$$

$$\begin{aligned} \frac{d^3y}{dx^3} &= a^2 \left[-\sin \left(\frac{2\pi}{2} + ax + b \right) \right] \frac{d}{dx} \left(\frac{2\pi}{2} + ax + b \right) \\ &= a^2 \cos \left(\frac{\pi}{2} + \frac{2\pi}{2} + ax + b \right) (a) \end{aligned}$$

$$\frac{d^3y}{dx^3} = a^3 \cos \left(\frac{3\pi}{2} + ax + b \right)$$

In general n^{th} order derivative will be

$$\frac{d^n y}{dx^n} = a^n \cos \left(\frac{n\pi}{2} + ax + b \right)$$

(vi) Let $y = e^{ax} \sin (bx + c)$

Differentiate *w. r. t. x*

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx} [e^{ax} \sin (bx + c)] = e^{ax} \frac{d}{dx} [\sin (bx + c)] + [\sin (bx + c)] \frac{d}{dx} (e^{ax}) \\ &= e^{ax} \cos (bx + c) \frac{d}{dx} (bx + c) + \sin (bx + c) \cdot e^{ax} \cdot \frac{d}{dx} (ax) \\ &= e^{ax} [b \cos (bx + c) + a \sin (bx + c)] \\ &= e^{ax} \sqrt{a^2 + b^2} \left[\frac{b}{\sqrt{a^2 + b^2}} \cos (bx + c) + \frac{a}{\sqrt{a^2 + b^2}} \sin (bx + c) \right]\end{aligned}$$

$$\text{Let } \frac{b}{\sqrt{a^2 + b^2}} = \sin \alpha, \frac{a}{\sqrt{a^2 + b^2}} = \cos \alpha, \alpha = \tan^{-1} \left(\frac{b}{a} \right) \quad \dots \text{(I)}$$

$$\frac{dy}{dx} = e^{ax} \sqrt{a^2 + b^2} [\sin \alpha \cdot \cos (bx + c) + \sin (bx + c) \cdot \cos \alpha]$$

$$\frac{dy}{dx} = e^{ax} (a^2 + b^2)^{\frac{1}{2}} \cdot \sin (bx + c + \alpha)$$

Differentiate *w. r. t. x*

$$\begin{aligned}\frac{d}{dx} \left(\frac{dy}{dx} \right) &= \frac{d}{dx} \left[e^{ax} (a^2 + b^2)^{\frac{1}{2}} \cdot \sin (bx + c + \alpha) \right] \\ &= (a^2 + b^2)^{\frac{1}{2}} \cdot \frac{d}{dx} [e^{ax} \cdot \sin (bx + c + \alpha)] \\ &= (a^2 + b^2)^{\frac{1}{2}} \left[e^{ax} \frac{d}{dx} [\sin (bx + c + \alpha)] + [\sin (bx + c + \alpha)] \frac{d}{dx} [e^{ax}] \right] \\ &= (a^2 + b^2)^{\frac{1}{2}} \left[e^{ax} \cos (bx + c + \alpha) \frac{d}{dx} (bx + c + \alpha) + \sin (bx + c + \alpha) \cdot e^{ax} \frac{d}{dx} (ax) \right] \\ &= e^{ax} (a^2 + b^2)^{\frac{1}{2}} [b \cos (bx + c + \alpha) + a \sin (bx + c + \alpha)] \\ &= e^{ax} (a^2 + b^2)^{\frac{1}{2}} \sqrt{a^2 + b^2} \left[\frac{b}{\sqrt{a^2 + b^2}} \cos (bx + c + \alpha) + \frac{a}{\sqrt{a^2 + b^2}} \sin (bx + c + \alpha) \right] \\ \frac{d^2 y}{dx^2} &= e^{ax} (a^2 + b^2)^{\frac{2}{2}} [\sin \alpha \cos (bx + c + \alpha) + \sin (bx + c + \alpha) \cos \alpha] \quad \dots \text{[from (I)]} \\ \frac{d^2 y}{dx^2} &= e^{ax} (a^2 + b^2)^{\frac{2}{2}} \cdot \sin (bx + c + 2\alpha)\end{aligned}$$

Similarly,

$$\frac{d^3 y}{dx^3} = e^{ax} (a^2 + b^2)^{\frac{3}{2}} \cdot \sin (bx + c + 3\alpha)$$

In general n^{th} order derivative will be

$$\frac{d^n y}{dx^n} = e^{ax} (a^2 + b^2)^{\frac{n}{2}} \cdot \sin (bx + c + n\alpha) \text{ where } \alpha = \tan^{-1} \left(\frac{b}{a} \right).$$

EXERCISE 1.5

(1) Find the second order derivative of the following :

(i) $2x^5 - 4x^3 - \frac{2}{x^2} - 9$ (ii) $e^{2x} \cdot \tan x$

(iii) $e^{4x} \cdot \cos 5x$ (iv) $x^3 \log x$

(v) $\log(\log x)$ (iv) x^x

(2) Find $\frac{d^2y}{dx^2}$ of the following :

(i) $x = a(\theta - \sin \theta), y = a(1 - \cos \theta)$

(ii) $x = 2at^2, y = 4at$

(iii) $x = \sin \theta, y = \sin^3 \theta$ when $\theta = \frac{\pi}{2}$

(iv) $x = a \cos \theta, y = b \sin \theta$ at $\theta = \frac{\pi}{4}$

(3) (i) If $x = at^2$ and $y = 2at$ then show that

$$xy \frac{d^2y}{dx^2} + a = 0$$

(ii) If $y = e^{m \tan^{-1} x}$, show that

$$(1 + x^2) \frac{d^2y}{dx^2} + (2x - m) \frac{dy}{dx} = 0$$

(iii) If $x = \cos t, y = e^{mt}$ show that

$$(1 - x^2) \frac{d^2y}{dx^2} - x \frac{dy}{dx} - m^2y = 0$$

(iv) If $y = x + \tan x$, show that

$$\cos^2 x \cdot \frac{d^2y}{dx^2} - 2y + 2x = 0$$

(v) If $y = e^{ax} \cdot \sin(bx)$, show that

$$y_2 - 2ay_1 + (a^2 + b^2)y = 0$$

(vi) If $\sec^{-1} \left(\frac{7x^3 - 5y^3}{7x^3 + 5y^3} \right) = m$,

show that $\frac{d^2y}{dx^2} = 0$.

(vii) If $2y = \sqrt{x+1} + \sqrt{x-1}$,

show that $4(x^2 - 1)y_2 + 4xy_1 - y = 0$.

(viii) If $y = \log(x + \sqrt{x^2 + a^2})^m$,

show that $(x^2 + a^2) \frac{d^2y}{dx^2} + x \frac{dy}{dx} = 0$

(ix) If $y = \sin(m \cos^{-1} x)$ then show that

$$(1 - x^2) \frac{d^2y}{dx^2} - x \frac{dy}{dx} + m^2y = 0$$

(x) If $y = \log(\log 2x)$, show that

$$x y_2 + y_1 (1 + x y_1) = 0.$$

(xi) If $x^2 + 6xy + y^2 = 10$, show that

$$\frac{d^2y}{dx^2} = \frac{80}{(3x + y)^3}.$$

(xii) If $x = a \sin t - b \cos t, y = a \cos t + b \sin t$,

show that $\frac{d^2y}{dx^2} = -\frac{x^2 + y^2}{y^3}$.

(4) Find the n^{th} derivative of the following :

(i) $(ax + b)^m$ (ii) $\frac{1}{x}$

(iii) e^{ax+b} (iv) a^{px+q}

(v) $\log(ax + b)$ (vi) $\cos x$

(vii) $\sin(ax + b)$ (viii) $\cos(3 - 2x)$

(ix) $\log(2x + 3)$

(x) $\frac{1}{3x - 5}$

(xi) $y = e^{ax} \cdot \cos(bx + c)$

(xii) $y = e^{8x} \cdot \cos(6x + 7)$



Let us Remember

✿ If a function $f(x)$ is differentiable at $x = a$ then it is continuous at $x = a$, but the converse is not true.

✿ **Chain Rule :** If y is differentiable function of u and u is differentiable function of x then y is differentiable function of x and $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

✿ If $y = f(x)$ is a differentiable of x such that the inverse function $x = f^{-1}(y)$ exists then

$$\frac{dy}{dx} = \frac{1}{\frac{dy}{dx}}, \quad \text{where } \frac{dy}{dx} \neq 0$$

✿ **Derivatives of Inverse Trigonometric functions :**

$f(x)$	$\sin^{-1} x$	$\cos^{-1} x$	$\tan^{-1} x$	$\cot^{-1} x$	$\sec^{-1} x$	$\operatorname{cosec}^{-1} x$
$f'(x)$	$\frac{1}{\sqrt{1-x^2}},$ $ x < 1$	$-\frac{1}{\sqrt{1-x^2}},$ $ x < 1$	$\frac{1}{1+x^2}$ $x \in \mathbb{R}$	$-\frac{1}{1+x^2}$ $x \in \mathbb{R}$	$\frac{1}{x\sqrt{x^2-1}}$ $ x < 1$	$-\frac{1}{x\sqrt{x^2-1}}$ $ x < 1$

✿ This is a simple shortcut to find the derivative of (function)^(function)

$$\frac{d}{dx} f^g = f^g \left[\frac{g}{f} \cdot f' + (\log f) \cdot g' \right]$$

✿ If $y = f(t)$ and $y = g(t)$ is a differentiable of t such that y is a function of x then

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}, \quad \text{where } \frac{dx}{dt} \neq 0$$

✿ Implicit function of the form $x^m y^n = (x + y)^{m+n}$, $m, n \in \mathbb{R}$ always have the first order derivative

$$\frac{dy}{dx} = \frac{y}{x} \text{ and second order derivative } \frac{d^2y}{dx^2} = 0$$

MISCELLANEOUS EXERCISE 1

(I) Choose the correct option from the given alternatives :

(1) Let $f(1) = 3, f'(1) = -\frac{1}{3}, g(1) = -4$ and $g'(1) = -\frac{8}{3}$. The derivative of $\sqrt{[f(x)]^2 + [g(x)]^2}$

w. r. t. x at $x = 1$ is

(A) $-\frac{29}{15}$

(B) $\frac{7}{3}$

(C) $\frac{31}{15}$

(D) $\frac{29}{15}$

- (2) If $y = \sec(\tan^{-1} x)$ then $\frac{dy}{dx}$ at $x = 1$, is equal to :
- (A) $\frac{1}{2}$ (B) 1 (C) $\frac{1}{\sqrt{2}}$ (D) $\sqrt{2}$
- (3) If $f(x) = \sin^{-1}\left(\frac{4^{x+\frac{1}{2}}}{1+2^{4x}}\right)$, which of the following is not the derivative of $f(x)$
- (A) $\frac{2 \cdot 4^x \log 4}{1+4^{2x}}$ (B) $\frac{4^{x+1} \log 2}{1+4^{2x}}$ (C) $\frac{4^{x+1} \log 4}{1+4^{4x}}$ (D) $\frac{2^{2(x+1)} \log 2}{1+2^{4x}}$
- (4) If $x^y = y^x$, then $\frac{dy}{dx} = \dots$
- (A) $\frac{x(x \log y - y)}{y(y \log x - x)}$ (B) $\frac{y(y \log x - x)}{x(x \log y - y)}$ (C) $\frac{y^2(1 - \log x)}{x^2(1 - \log y)}$ (D) $\frac{y(1 - \log x)}{x(1 - \log y)}$
- (5) If $y = \sin(2 \sin^{-1} x)$, then $\frac{dy}{dx} = \dots$
- (A) $\frac{2 - 4x^2}{\sqrt{1-x^2}}$ (B) $\frac{2 + 4x^2}{\sqrt{1-x^2}}$ (C) $\frac{4x^2 - 1}{\sqrt{1-x^2}}$ (D) $\frac{1 - 2x^2}{\sqrt{1-x^2}}$
- (6) If $y = \tan^{-1}\left(\frac{x}{1+\sqrt{1-x^2}}\right) + \sin\left[2 \tan^{-1}\left(\sqrt{\frac{1-x}{1+x}}\right)\right]$, then $\frac{dy}{dx} = \dots$
- (A) $\frac{x}{\sqrt{1-x^2}}$ (B) $\frac{1-2x}{\sqrt{1-x^2}}$ (C) $\frac{1-2x}{2\sqrt{1-x^2}}$ (D) $\frac{1-2x^2}{\sqrt{1-x^2}}$
- (7) If y is a function of x and $\log(x+y) = 2xy$, then the value of $y'(0) = \dots$
- (A) 2 (B) 0 (C) -1 (D) 1
- (8) If g is the inverse of a function f and $f'(x) = \frac{1}{1+x^7}$, then the value of $g'(x)$ is equal to :
- (A) $1+x^7$ (B) $\frac{1}{1+[g(x)]^7}$ (C) $1+[g(x)]^7$ (D) $7x^6$
- (9) If $x\sqrt{y+1} + y\sqrt{x+1} = 0$ and $x \neq y$ then $\frac{dy}{dx} = \dots$
- (A) $\frac{1}{(1+x)^2}$ (B) $-\frac{1}{(1+x)^2}$ (C) $(1+x)^2$ (D) $-\frac{x}{x+1}$
- (10) If $y = \tan^{-1}\left(\sqrt{\frac{a-x}{a+x}}\right)$, where $-a < x < a$ then $\frac{dy}{dx} = \dots$
- (A) $\frac{x}{\sqrt{a^2-x^2}}$ (B) $\frac{a}{\sqrt{a^2-x^2}}$ (C) $-\frac{1}{2\sqrt{a^2-x^2}}$ (D) $\frac{1}{2\sqrt{a^2-x^2}}$

(11) If $x = a(\cos \theta + \theta \sin \theta)$, $y = a(\sin \theta - \theta \cos \theta)$ then $\left[\frac{d^2y}{dx^2} \right]_{\theta = \frac{\pi}{4}} = \dots$

- (A) $\frac{8\sqrt{2}}{a\pi}$ (B) $-\frac{8\sqrt{2}}{a\pi}$ (C) $\frac{a\pi}{8\sqrt{2}}$ (D) $\frac{4\sqrt{2}}{a\pi}$

(12) If $y = a \cos(\log x)$ and $A \frac{d^2y}{dx^2} + B \frac{dy}{dx} + C = 0$, then the values of A, B, C are ...

- (A) $x^2, -x, -y$ (B) x^2, x, y (C) $x^2, x, -y$ (D) $x^2, -x, y$

(II) Solve the following :

- (1) $f(x) = -x,$ for $-2 \leq x < 0$ $g(x) = 6 - 3x,$ for $0 \leq x \leq 2$
 $= 2x,$ for $0 \leq x \leq 2$ $= \frac{2x - 4}{3},$ for $2 < x \leq 7$
 $= \frac{18 - x}{4},$ for $2 < x \leq 7$

Let $u(x) = f[g(x)]$, $v(x) = g[f(x)]$ and $w(x) = g[g(x)]$.

Find each derivative at $x = 1$, if it exists i.e. find $u'(1)$, $v'(1)$ and $w'(1)$. if it doesn't exist then explain why ?

- (2) The values of $f(x), g(x), f'(x)$ and $g'(x)$ are given in the following table.

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
-1	3	2	-3	4
2	2	-1	-5	-4

Match the following.

A Group - Function	B Group - Derivative
(A) $\frac{d}{dx} [f(g(x))] \text{ at } x = -1$	1. -16
(B) $\frac{d}{dx} [g(f(x) - 1)] \text{ at } x = -1$	2. 20
(C) $\frac{d}{dx} [f(f(x) - 3)] \text{ at } x = 2$	3. -20
(D) $\frac{d}{dx} [g(g(x))] \text{ at } x = 2$	4. 15
	5. 12

- (3) Suppose that the functions f and g and their derivatives with respect to x have the following values at $x = 0$ and $x = 1$.

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
0	1	1	5	$\frac{1}{3}$
1	3	-4	$-\frac{1}{3}$	$-\frac{8}{3}$

- (i) The derivative of $f[g(x)]$ w. r. t. x at $x = 0$ is
- (ii) The derivative of $g[f(x)]$ w. r. t. x at $x = 0$ is
- (iii) The value of $\left[\frac{d}{dx} [x^{10} + f(x)]^{-2} \right]_{x=1}$ is
- (iv) The derivative of $f[(x + g(x))]$ w. r. t. x at $x = 0$ is

