

7. LIMITS



Let's study.

- Definition of Limit
- Algebra of Limits
- Evaluation of Limits
 - Direct Method
 - Factorization Method
 - Rationalization Method
- Limits of Exponential and Logarithmic Functions

Introduction:

Calculus is one of the important branches of Mathematics. The concept of limit of a function is a fundamental concept in calculus.

Let's understand

Meaning of $x \rightarrow a$:

When x takes the values gradually nearer to a , we say that ' x tends to a '. This is symbolically written as ' $x \rightarrow a$ '.

' $x \rightarrow a$ ' implies that $x \neq a$ and hence $(x-a) \neq 0$

Limit of a function :

Let us understand the concept by an example.

Consider the function $f(x) = x + 3$

Take the value of x very close to 3, but not equal to 3; and observe the values of $f(x)$.

	x approaches 3 from left				
x	2.5	2.6	...	2.9	2.99
$f(x)$	5.5	5.6	...	5.9	5.99

	x approaches 3 from right				
x	3.6	3.5	...	3.1	3.01
$f(x)$	6.6	6.5	...	6.1	6.01

From the table we observe that as $x \rightarrow 3$ from either side. $f(x) \rightarrow 6$.

This idea can be expressed by saying that the limiting value of $f(x)$ is 6 when x approaches to 3.

This is symbolically written as,

$$\lim_{x \rightarrow 3} f(x) = 6$$

i.e. $\lim_{x \rightarrow 3} (x+3) = 6$

Thus, limit of the function, $f(x) = x + 3$ as $x \rightarrow 3$ is the value of the function at $x = 3$.



Let's learn.

7.1 DEFINITION OF LIMIT OF A FUNCTION:

A function $f(x)$ is said to have the limit l as x tends to a , if for every $\epsilon > 0$ we can find $\delta > 0$ such that, $|f(x) - l| < \epsilon$ whenever $0 < |x-a| < \delta$ and ' l ' is a finite real number.

We are going to study the limit of a rational

function $\frac{P(x)}{Q(x)}$ as $x \rightarrow a$.

Here $P(x)$ and $Q(x)$ are polynomials in x .

We get three different cases.

- (1) $Q(a) \neq 0$,
- (2) $Q(a) = 0$ and $P(a) = 0$
- (3) $Q(a) = 0$ and $P(a) \neq 0$

In case (1) $\lim_{x \rightarrow a} \frac{P(x)}{Q(x)} = \frac{P(a)}{Q(a)}$.

Because as $x \rightarrow a$, $P(x) \rightarrow P(a)$ and $Q(x) \rightarrow Q(a)$

In Case (2) $x - a$ is a factor of $P(x)$ as well as $Q(x)$ so we have express $P(x)$ and $Q(x)$ as $P(x) = (x - a) P_1(x)$ and $Q(x) = (x - a) Q_1(x)$

$$\text{Now } \frac{P(x)}{Q(x)} = \frac{(x-a)P_1(x)}{(x-a)Q_1(x)} = \frac{P_1(x)}{Q_1(x)}$$

Note that

$(x-a) \neq 0$ so we can cancel the factor.

In case (3) $Q(a) = 0$ and $P(a) \neq 0$,

$$\lim_{x \rightarrow a} \frac{P(x)}{Q(x)} \text{ does not exist.}$$

7.1.1 One Sided Limit: You are aware of the fact that when $x \rightarrow a$; x approaches a in two directions; which can lead to two limits, known as left hand limit and right hand limit.

7.1.2 Right hand Limit : If given $\epsilon > 0$ (however small), there exists $\delta > 0$ such that $|f(x) - l_1| < \epsilon$ for all x with $a < x < a + \delta$ then $\lim_{x \rightarrow a^+} f(x) = l_1$

7.1.3 Left hand Limit : If given $\epsilon > 0$ (however small), there exists $\delta > 0$ such that for $|f(x) - l_2| < \epsilon$ all x with $a - \delta < x < a$ then $\lim_{x \rightarrow a^-} f(x) = l_2$

Example:

Find left hand limit and right hand limit for the following example.

$$f(x) = \begin{cases} 3x+1 & \text{if } x < 1 \\ 7x^2-3 & \text{if } x \geq 1 \end{cases}$$

To compute, $\lim_{x \rightarrow 1^+} f(x)$, we use the definition for f which applies to $x \geq 1$:

$$\lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} (7x^2 - 3) = 4$$

Likewise, to compute $\lim_{x \rightarrow 1^-} f(x)$, we use the definition for f which applies to $x < 1$:

$$\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^-} (3x + 1) = 4$$

Since left and right-hand limits are equal,

$$\lim_{x \rightarrow 1} f(x) = 4$$

7.1.4 Existence of a limit of a function at a point $x = a$

If $\lim_{x \rightarrow a^+} f(x) = \lim_{x \rightarrow a^-} f(x) = l$, then limit of the function $f(x)$ as $x \rightarrow a$ exists and its value is l . If

$\lim_{x \rightarrow a^+} f(x) \neq \lim_{x \rightarrow a^-} f(x)$ then $\lim_{x \rightarrow a} f(x)$ does not exist.



7.2 ALGEBRA OF LIMITS:

Let $f(x)$ and $g(x)$ be two functions such that $\lim_{x \rightarrow a} f(x) = l$ and $\lim_{x \rightarrow a} g(x) = m$, then

- $\lim_{x \rightarrow a} [f(x) \pm g(x)] = \lim_{x \rightarrow a} f(x) \pm \lim_{x \rightarrow a} g(x) = l \pm m$
- $\lim_{x \rightarrow a} [f(x) \times g(x)] = \lim_{x \rightarrow a} f(x) \times \lim_{x \rightarrow a} g(x) = l \times m$
- $\lim_{x \rightarrow a} [k f(x)] = k \lim_{x \rightarrow a} f(x) = kl$, where 'k' is a constant
- $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} g(x)} = \frac{l}{m}$ provided $m \neq 0$

7.3 EVALUATION OF LIMITS :

Direct Method : In some cases $\lim_{x \rightarrow a} f(x)$ can be obtained by just substitution of x by a in $f(x)$

SOLVED EXAMPLES

Ex. 1) $\lim_{r \rightarrow 1} \left(\frac{4}{3} \pi r^2 \right) = \frac{4}{3} \pi \lim_{r \rightarrow 1} (r^2) = \frac{4}{3} \pi (1)^2 = \frac{4}{3} \pi$

Ex. 2) $\lim_{y \rightarrow 2} [(y^2 - 3)(y + 2)]$
 $= \lim_{y \rightarrow 2} (y^2 - 3) \lim_{y \rightarrow 2} (y + 2)$
 $= (2^2 - 3)(2 + 2) = (4 - 3)(4) = 1 \times 4 = 4$

Ex. 3) $\lim_{x \rightarrow 3} \left(\frac{\sqrt{6+x} - \sqrt{7-x}}{x} \right)$
 $= \frac{\lim_{x \rightarrow 3} (\sqrt{6+x}) - \lim_{x \rightarrow 3} (\sqrt{7-x})}{\lim_{x \rightarrow 3} (x)}$
 $= \frac{\sqrt{6+3} - \sqrt{7-3}}{3}$
 $= \frac{\sqrt{9} - \sqrt{4}}{3}$
 $= \frac{3-2}{3} = \frac{1}{3}$

Ex. 4) Discuss the limit of the following function as x tends to 3 if

$$f(x) = \begin{cases} x^2 + x + 1, & 2 \leq x \leq 3 \\ 2x + 1, & 3 < x \leq 4 \end{cases}$$

Solution: we use the concept of left hand limit and right hand limit, to discuss the existence of limit as $x \rightarrow 3$

Note : In both cases x takes only positive values.

For the interval $2 \leq x \leq 3$; $f(x) = x^2 + x + 1$

$$\begin{aligned} \therefore \lim_{x \rightarrow 3^-} f(x) &= \lim_{x \rightarrow 3^-} (x^2 + x + 1) \\ &= (3)^2 + 3 + 1 \\ &= 9 + 3 + 1 = 13 \text{ -----(I)} \end{aligned}$$

Similarly for the interval $3 < x \leq 4$;

$$f(x) = 2x + 1$$

$$\begin{aligned} \lim_{x \rightarrow 3^+} f(x) &= \lim_{x \rightarrow 3^+} (2x + 1) = (2 \times 3) + 1 = 6 + 1 \\ &= 7 \text{ -----(II)} \end{aligned}$$

From (I) and (II), $\lim_{x \rightarrow 3^-} f(x) \neq \lim_{x \rightarrow 3^+} f(x)$

$\therefore \lim_{x \rightarrow 3} f(x)$ does not exist.

Theorem: $\lim_{x \rightarrow a} \frac{x^n - a^n}{x - a} = n(a^{n-1})$, for $n \in \mathbb{Q}$.

SOLVED EXAMPLES

Ex. 1) $\lim_{x \rightarrow 4} \left[\frac{x^n - 4^n}{x - 4} \right] = 48$ and $n \in \mathbb{N}$, find n .

Solution: Given $\lim_{x \rightarrow 4} \left[\frac{x^n - 4^n}{x - 4} \right] = 48$

$$\therefore n(4)^{n-1} = 48 = 3 \times 16 = 3(4)^2$$

$$\therefore n(4)^{n-1} = 3(4)^{3-1} \dots \text{by observation}$$

$$\therefore n = 3.$$

Ex. 2) Evaluate $\lim_{x \rightarrow 1} \left[\frac{2x - 2}{\sqrt[3]{26 + x} - 3} \right]$

Solution: Put $26 + x = t^3$, $\therefore x = t^3 - 26$

As $x \rightarrow 1$, $t \rightarrow 3$

$$\therefore \lim_{x \rightarrow 1} \left[\frac{2x - 2}{\sqrt[3]{26 + x} - 3} \right]$$

$$= \lim_{t \rightarrow 3} \left[\frac{2(t^3 - 26) - 2}{\sqrt[3]{t^3} - 3} \right]$$

$$= \lim_{t \rightarrow 3} \left[\frac{t^3 - 3^3}{t - 3} \right]$$

$$\text{As } \lim_{x \rightarrow a} \left[\frac{x^n - a^n}{x - a} \right] = na^{n-1}$$

$$\begin{aligned}
 &= 2 \times 3(3)^{3-1} \\
 &= 2 \times 3^3 = 2 \times 27 \\
 &= 54
 \end{aligned}$$

$$3. \lim_{z \rightarrow a} \left[\frac{(z+2)^{\frac{3}{2}} - (a+2)^{\frac{3}{2}}}{z-a} \right]$$

$$4. \lim_{x \rightarrow 5} \left[\frac{x^3 - 125}{x^2 - 25} \right]$$

EXERCISE 7.1

Q.I Evaluate the Following limits :

$$1. \lim_{x \rightarrow 3} \left[\frac{\sqrt{x+6}}{x} \right]$$

$$2. \lim_{x \rightarrow 2} \left[\frac{x^{-3} - 2^{-3}}{x-2} \right]$$

$$3. \lim_{x \rightarrow 5} \left[\frac{x^3 - 125}{x^5 - 3125} \right]$$

$$4. \text{ If } \lim_{x \rightarrow 1} \left[\frac{x^4 - 1}{x-1} \right] = \lim_{x \rightarrow a} \left[\frac{x^3 - a^3}{x-a} \right] \text{ find all possible values of } a.$$

Q.II Evaluate the Following limits :

$$1. \lim_{x \rightarrow 7} \left[\frac{(\sqrt[3]{x} - \sqrt[3]{7})(\sqrt[3]{x} + \sqrt[3]{7})}{x-7} \right]$$

$$2. \text{ If } \lim_{x \rightarrow 5} \left[\frac{x^k - 5^k}{x-5} \right] = 500 \text{ find all possible values of } k.$$

$$3. \lim_{x \rightarrow 0} \left[\frac{(1-x)^8 - 1}{(1-x)^2 - 1} \right]$$

Q.III Evaluate the Following limits :

$$1. \lim_{x \rightarrow 0} \left[\frac{\sqrt[3]{1+x} - \sqrt{1+x}}{x} \right]$$

$$2. \lim_{y \rightarrow 1} \left[\frac{2y-2}{\sqrt[3]{7+y}-2} \right]$$



Let's learn.

7.4 FACTORIZATION METHOD :

Consider the problem of evaluating,

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} \text{ where } g(a) \neq 0.$$

SOLVED EXAMPLES

$$\text{Ex. 1) Evaluate } \lim_{z \rightarrow 3} \left[\frac{z(2z-3)-9}{z^2-4z+3} \right]$$

Solution: If we substitute $z = 3$ in numerator and denominator,

$$\text{we get } z(2z-3) - 9 = 0 \text{ and } z^2 - 4z + 3 = 0.$$

So $(z-3)$ is a factor in the numerator and denominator.

$$\therefore \lim_{z \rightarrow 3} \left[\frac{z(2z-3)-9}{z^2-4z+3} \right]$$

$$= \lim_{z \rightarrow 3} \left[\frac{2z^2 - 3z - 9}{z^2 - 4z + 3} \right]$$

$$= \lim_{z \rightarrow 3} \left[\frac{(z-3)(2z+3)}{(z-3)(z-1)} \right]$$

$$= \lim_{z \rightarrow 3} \left[\frac{(2z+3)}{(z-1)} \right] \quad \because (z-3 \neq 0)$$

$$= \frac{2(3)+3}{3-1}$$

$$= \frac{9}{2}.$$

Ex. 2) Evaluate $\lim_{x \rightarrow 4} \left[\frac{(x^3 - 8x^2 + 16x)^9}{(x^2 - x - 12)^{18}} \right]$

Solution : $\lim_{x \rightarrow 4} \left[\frac{[x(x-4)^2]^9}{(x-4)^{18} (x+3)^{18}} \right]$
 $= \lim_{x \rightarrow 4} \left[\frac{(x-4)^{18} x^9}{(x-4)^{18} (x+3)^{18}} \right]$
 $= \lim_{x \rightarrow 4} \left[\frac{x^9}{(x+3)^{18}} \right] \because (x-4) \neq 0$
 $= \frac{4^9}{7^{18}}$

Ex. 3) Evaluate $\lim_{x \rightarrow 1} \left[\frac{1}{x-1} + \frac{2}{1-x^2} \right]$

Solution : $\lim_{x \rightarrow 1} \left[\frac{1}{x-1} + \frac{2}{(1-x)(x+1)} \right]$
 $= \lim_{x \rightarrow 1} \left[\frac{1}{x-1} - \frac{2}{(x-1)(x+1)} \right]$
 $= \lim_{x \rightarrow 1} \left[\frac{1+x-2}{(x-1)(x+1)} \right]$
 $= \lim_{x \rightarrow 1} \left[\frac{x-1}{(x-1)(x+1)} \right]$

Since $(x \rightarrow 1), (x-1 \neq 0)$

$$= \lim_{x \rightarrow 1} \left[\frac{1}{(x+1)} \right]$$

$$= \frac{1}{2}$$

Ex. 4) Evaluate $\lim_{x \rightarrow 1} \left[\frac{x^3 + x^2 - 5x + 3}{x^2 - 1} \right]$

Solution : In this case $(x-1)$ is a factor of the numerator and denominator.

To find another factor we use synthetic division, Numerator: $x^3 + x^2 - 5x + 3$

1	1	1	-5	3
		1	2	-3
	1	2 (=1+1)	-3 (= -5+2)	0 (= -3+3)

$$\therefore x^3 + x^2 - 5x + 3 = (x-1)(x^2 + 2x - 3)$$

$$\text{Denominator: } x^2 - 1 = (x+1)(x-1)$$

$$= \lim_{x \rightarrow 1} \left[\frac{x^3 + x^2 - 5x + 3}{x^2 - 1} \right]$$

$$= \lim_{x \rightarrow 1} \left[\frac{(x-1)(x^2 + 2x - 3)}{(x+1)(x-1)} \right]$$

$$= \lim_{x \rightarrow 1} \left[\frac{(x^2 + 2x - 3)}{(x+1)} \right] \begin{matrix} (x \rightarrow 1) \\ x \neq 1 \\ x-1 \neq 0 \end{matrix}$$

$$= \frac{1+2-3}{1+1}$$

$$= 0$$

Ex. 5) $\lim_{x \rightarrow 1} \left[\frac{\frac{1}{x} - 1}{x-1} \right]$

$$= \lim_{x \rightarrow 1} \left[\frac{(1-x)}{(x-1) \times x} \right]$$

$$= \lim_{x \rightarrow 1} \left[\frac{-(x-1)}{(x-1) \times x} \right]$$

since $x-1 \neq 0$

$$= \lim_{x \rightarrow 1} \left[\frac{-1}{x} \right] = -\lim_{x \rightarrow 1} \left[\frac{1}{x} \right] = -\frac{1}{1}$$

$$= -1$$

EXERCISE 7.2

Q.I Evaluate the following limits :

1. $\lim_{z \rightarrow 2} \left[\frac{z^2 - 5z + 6}{z^2 - 4} \right]$

2. $\lim_{x \rightarrow -3} \left[\frac{x + 3}{x^2 + 4x + 3} \right]$

3. $\lim_{y \rightarrow 0} \left[\frac{5y^3 + 8y^2}{3y^4 - 16y^2} \right]$

4. $\lim_{x \rightarrow -2} \left[\frac{-2x - 4}{x^3 + 2x^2} \right]$

Q.II Evaluate the following limits :

1. $\lim_{u \rightarrow 1} \left[\frac{u^4 - 1}{u^3 - 1} \right]$

2. $\lim_{x \rightarrow 3} \left[\frac{1}{x - 3} - \frac{9x}{x^3 - 27} \right]$

3. $\lim_{x \rightarrow 2} \left[\frac{x^3 - 4x^2 + 4x}{x^2 - 1} \right]$

Q.III Evaluate the following limits :

1. $\lim_{x \rightarrow -2} \left[\frac{x^7 + x^5 + 160}{x^3 + 8} \right]$

2. $\lim_{y \rightarrow \frac{1}{2}} \left[\frac{1 - 8y^3}{y - 4y^3} \right]$

3. $\lim_{v \rightarrow \sqrt{2}} \left[\frac{v^2 + v\sqrt{2} - 4}{v^2 - 3v\sqrt{2} + 4} \right]$

4. $\lim_{x \rightarrow 3} \left[\frac{x^2 + 2x - 15}{x^2 - 5x + 6} \right]$



Let's learn.

7.5 RATIONALIZATION METHOD :

If the function in the limit involves a square root, it may be possible to simplify the expression by multiplying and dividing by the conjugate. This method uses the algebraic identity.

Here, we do the following steps:

Step 1. **Rationalize the factor containing square root.**

Step 2. **Simplify.**

Step 3. **Put the value of x and get the required result.**

SOLVED EXAMPLES

Ex. 1) Evaluate $\lim_{z \rightarrow 0} \left[\frac{(b+z)^{\frac{1}{2}} - (b-z)^{\frac{1}{2}}}{z} \right]$

Solution: $\lim_{z \rightarrow 0} \left[\frac{(b+z)^{\frac{1}{2}} - (b-z)^{\frac{1}{2}}}{z} \right]$

$$= \lim_{z \rightarrow 0} \left[\frac{\sqrt{b+z} - \sqrt{b-z}}{z} \right]$$

$$= \lim_{z \rightarrow 0} \left[\frac{\sqrt{b+z} - \sqrt{b-z}}{z} \times \frac{\sqrt{b+z} + \sqrt{b-z}}{\sqrt{b+z} + \sqrt{b-z}} \right]$$

$$= \lim_{z \rightarrow 0} \left[\frac{(\sqrt{b+z})^2 - (\sqrt{b-z})^2}{z} \times \frac{1}{\sqrt{b+z} + \sqrt{b-z}} \right]$$

$$= \lim_{z \rightarrow 0} \left[\frac{(b+z) - (b-z)}{z} \times \frac{1}{\sqrt{b+z} + \sqrt{b-z}} \right]$$

$$= \lim_{z \rightarrow 0} \left[\frac{2z}{z} \times \frac{1}{\sqrt{b+z} + \sqrt{b-z}} \right]$$

$$\begin{aligned}
&= \lim_{z \rightarrow 0} \left[\frac{2}{\sqrt{b+z} + \sqrt{b-z}} \right] \\
&= \frac{2}{\sqrt{b+0} + \sqrt{b-0}} \\
&= \frac{2}{2\sqrt{b}} \\
&= \frac{1}{\sqrt{b}}
\end{aligned}$$

Ex. 2) Evaluate $\lim_{x \rightarrow 0} \left[\frac{1 - \sqrt{x^2 + 1}}{x^2} \right]$

Solution :

$$\begin{aligned}
&\lim_{x \rightarrow 0} \left[\frac{1 - \sqrt{x^2 + 1}}{x^2} \right] \\
&= \lim_{x \rightarrow 0} \left[\frac{1 - \sqrt{x^2 + 1} (1 + \sqrt{x^2 + 1})}{x^2 (1 + \sqrt{x^2 + 1})} \right] \\
&= \lim_{x \rightarrow 0} \left[\frac{1 - (x^2 + 1)}{x^2} \frac{1}{1 + \sqrt{x^2 + 1}} \right] \\
&= \lim_{x \rightarrow 0} \left[\frac{(-x^2)}{x^2} \frac{1}{1 + \sqrt{x^2 + 1}} \right] \\
&= \lim_{x \rightarrow 0} \left[\frac{-1}{1 + \sqrt{x^2 + 1}} \right] \\
&= \frac{-1}{1 + 1} \\
&= \frac{-1}{2}
\end{aligned}$$

EXERCISE 7.3

Q.I Evaluate the following limits :

1. $\lim_{x \rightarrow 0} \left[\frac{\sqrt{6+x+x^2} - \sqrt{6}}{x} \right]$
2. $\lim_{y \rightarrow 0} \left[\frac{\sqrt{1-y^2} - \sqrt{1+y^2}}{y^2} \right]$
3. $\lim_{x \rightarrow 2} \left[\frac{\sqrt{2+x} - \sqrt{6-x}}{\sqrt{x} - \sqrt{2}} \right]$

Q.II Evaluate the following limits :

1. $\lim_{x \rightarrow a} \left[\frac{\sqrt{a+2x} - \sqrt{3x}}{\sqrt{3a+x} - 2\sqrt{x}} \right]$
2. $\lim_{x \rightarrow 2} \left[\frac{x^2 - 4}{\sqrt{x+2} - \sqrt{3x-2}} \right]$

Q.III Evaluate the Following limits :

1. $\lim_{x \rightarrow 1} \left[\frac{x^2 + x\sqrt{x} - 2}{x-1} \right]$
2. $\lim_{x \rightarrow 0} \left[\frac{\sqrt{1+x^2} - \sqrt{1+x}}{\sqrt{1+x^3} - \sqrt{1+x}} \right]$
3. $\lim_{x \rightarrow 4} \left[\frac{x^2 + x - 20}{\sqrt{3x+4} - 4} \right]$
4. $\lim_{x \rightarrow 2} \left[\frac{x^3 - 8}{\sqrt{x+2} - \sqrt{3x-2}} \right]$

Q.IV Evaluate the Following limits :

1. $\lim_{y \rightarrow 2} \left[\frac{2-y}{\sqrt{3-y} - 1} \right]$
2. $\lim_{z \rightarrow 4} \left[\frac{3 - \sqrt{5+z}}{1 - \sqrt{5-z}} \right]$

7.6 LIMITS OF EXPONENTIAL AND LOGARITHMIC FUNCTIONS :

We use the following results without proof.

$$1. \lim_{x \rightarrow 0} \frac{a^x - 1}{x} = \log a, \quad a > 0,$$

$$2. \lim_{x \rightarrow 0} \frac{e^x - 1}{x} = \log e = 1$$

$$3. \lim_{x \rightarrow 0} [1 + x]^{\frac{1}{x}} = e$$

$$4. \lim_{x \rightarrow 0} \frac{\log(1+x)}{x} = 1$$

SOLVED EXAMPLES

Ex.1) Evaluate : $\lim_{x \rightarrow 0} \left[\frac{7^x - 1}{x} \right]$

Solution : $\lim_{x \rightarrow 0} \left[\frac{7^x - 1}{x} \right]$
 $= \log 7$

Ex.2) Evaluate : $\lim_{x \rightarrow 0} \left[\frac{5^x - 3^x}{x} \right]$

Solutions : $\lim_{x \rightarrow 0} \left[\frac{5^x - 3^x}{x} \right]$

$$\begin{aligned} &= \lim_{x \rightarrow 0} \left[\frac{5^x - 1 - 3^x + 1}{x} \right] \\ &= \lim_{x \rightarrow 0} \left[\frac{(5^x - 1)}{x} - \frac{(3^x - 1)}{x} \right] \\ &= \lim_{x \rightarrow 0} \left[\frac{(5^x - 1)}{x} \right] - \lim_{x \rightarrow 0} \left[\frac{(3^x - 1)}{x} \right] \\ &= \log 5 - \log 3 \\ &= \log \left(\frac{5}{3} \right) \end{aligned}$$

Ex.3) Evaluate : $\lim_{x \rightarrow 0} \left[1 + \frac{5}{6}x \right]^{\frac{1}{x}}$

Solutions : $\lim_{x \rightarrow 0} \left[1 + \frac{5}{6}x \right]^{\frac{1}{x}}$
 $= \lim_{x \rightarrow 0} \left[\left(1 + \frac{5}{6}x \right)^{\frac{1}{\frac{5}{6}x}} \right]^{\frac{5}{6}}$
 $= e^{\frac{5}{6}}$

Ex.4) Evaluate : $\lim_{x \rightarrow 0} \left[\frac{\log(1+4x)}{x} \right]$

Solutions : $\lim_{x \rightarrow 0} \left[\frac{\log(1+4x)}{x} \right]$
 $= \lim_{x \rightarrow 0} \left[\frac{\log(1+4x)}{4x} \times 4 \right]$
 $= 4 \times 1$
 $= 4$

Ex.5) Evaluate :

$$\lim_{x \rightarrow 0} \left[\frac{8^x - 4^x - 2^x + 1}{x^2} \right]$$

Solutions : Given $\lim_{x \rightarrow 0} \left[\frac{8^x - 4^x - 2^x + 1}{x^2} \right]$

$$\begin{aligned} &= \lim_{x \rightarrow 0} \left[\frac{(4 \times 2)^x - 4^x - 2^x + 1}{x^2} \right] \\ &= \lim_{x \rightarrow 0} \left[\frac{4^x \cdot 2^x - 4^x - 2^x + 1}{x^2} \right] \\ &= \lim_{x \rightarrow 0} \left[\frac{4^x (2^x - 1) - (2^x - 1)}{x^2} \right] \end{aligned}$$

$$\begin{aligned}
&= \lim_{x \rightarrow 0} \left[\frac{(2^x - 1) \cdot (4^x - 1)}{x^2} \right] \\
&= \lim_{x \rightarrow 0} \left[\frac{(2^x - 1)}{x} \right] \times \lim_{x \rightarrow 0} \left[\frac{(4^x - 1)}{x} \right] \\
&= (\log 2) (\log 4)
\end{aligned}$$

$$4) \lim_{x \rightarrow 2} \left[\frac{3^{\frac{x}{2}} - 3}{3^x - 9} \right]$$

IV] Evaluate the following :

$$1) \lim_{x \rightarrow 0} \left[\frac{(25)^x - 2(5)^x + 1}{x^2} \right]$$

$$2) \lim_{x \rightarrow 0} \left[\frac{(49)^x - 2(35)^x + (25)^x}{x^2} \right]$$

EXERCISE 7.4

I] Evaluate the following :

$$1) \lim_{x \rightarrow 0} \left[\frac{9^x - 5^x}{4^x - 1} \right]$$

$$2) \lim_{x \rightarrow 0} \left[\frac{5^x + 3^x - 2^x - 1}{x} \right]$$

$$3) \lim_{x \rightarrow 0} \left[\frac{\log(2+x) - \log(2-x)}{x} \right]$$

II] Evaluate the following :

$$1) \lim_{x \rightarrow 0} \left[\frac{3^x + 3^{-x} - 2}{x^2} \right]$$

$$2) \lim_{x \rightarrow 0} \left[\frac{3+x}{3-x} \right]^{\frac{1}{x}}$$

$$3) \lim_{x \rightarrow 0} \left[\frac{\log(3-x) - \log(3+x)}{x} \right]$$

III] Evaluate the following :

$$1) \lim_{x \rightarrow 0} \left[\frac{a^{3x} - b^{2x}}{\log(1+4x)} \right]$$

$$2) \lim_{x \rightarrow 0} \left[\frac{(2^x - 1)^2}{(3^x - 1) \cdot \log(1+x)} \right]$$

$$3) \lim_{x \rightarrow 0} \left[\frac{15^x - 5^x - 3^x + 1}{x^2} \right]$$



Let's learn.

Some Standard Results

$$1. \lim_{x \rightarrow a} k = k, \text{ where } k \text{ is a constant}$$

$$2. \lim_{x \rightarrow a} x = a$$

$$3. \lim_{x \rightarrow a} x^n = a^n$$

$$4. \lim_{x \rightarrow a} \sqrt[x]{x} = \sqrt[x]{a}$$

$$5. \text{ If } p(x) \text{ is a polynomial, then } \lim_{x \rightarrow a} p(x) = p(a)$$

$$6. \lim_{x \rightarrow a} \frac{x^n - a^n}{x - a} = n(a^{n-1}), \text{ for } n \in \mathbb{Q}$$

$$7. \lim_{x \rightarrow 0} \left(\frac{e^x - 1}{x} \right) = \log e = 1$$

$$8. \lim_{x \rightarrow 0} \frac{\log(1+x)}{x} = 1$$

MISCELLANEOUS EXERCISE - 7

I. If $\lim_{x \rightarrow 2} \frac{x^n - 2^n}{x - 2} = 80$ then find the value of n .

II. Evaluate the following Limits.

$$1) \lim_{x \rightarrow a} \frac{(x+2)^{\frac{5}{3}} - (a+2)^{\frac{5}{3}}}{x - a}$$

$$2) \lim_{x \rightarrow 2} \frac{(1+x)^n - 1}{x}$$

$$3) \lim_{x \rightarrow 2} \left[\frac{(x-2)}{2x^2 - 7x + 6} \right]$$

$$4) \lim_{x \rightarrow 1} \left[\frac{x^3 - 1}{x^2 + 5x - 6} \right]$$

$$5) \lim_{x \rightarrow 3} \left[\frac{x-3}{\sqrt{x-2} - \sqrt{4-x}} \right]$$

$$6) \lim_{x \rightarrow 4} \left[\frac{3 - \sqrt{5+x}}{1 - \sqrt{5-x}} \right]$$

$$7) \lim_{x \rightarrow 0} \left[\frac{5^x - 1}{x} \right]$$

$$8) \lim_{x \rightarrow 0} \left(1 + \frac{x}{5} \right)^{\frac{1}{x}}$$

$$9) \lim_{x \rightarrow 0} \left[\frac{\log(1+9x)}{x} \right]$$

$$10) \lim_{x \rightarrow 0} \frac{(1-x)^5 - 1}{(1-x)^3 - 1}$$

$$11) \text{ Evaluate : } \lim_{x \rightarrow 0} \left[\frac{a^x + b^x + c^x - 3}{x} \right]$$

$$12) \lim_{x \rightarrow 0} \frac{e^x + e^{-x} - 2}{x^2}$$

$$13) \lim_{x \rightarrow 0} \left[\frac{x(6^x - 3^x)}{(2^x - 1) \cdot \log(1+x)} \right]$$

$$14) \lim_{x \rightarrow 0} \left[\frac{a^{3x} - a^{2x} - a^x + 1}{x^2} \right]$$

$$15) \lim_{x \rightarrow 0} \left[\frac{(5^x - 1)^2}{x \cdot \log(1+x)} \right]$$

$$16) \lim_{x \rightarrow 0} \left[\frac{a^{4x} - 1}{b^{2x} - 1} \right]$$

$$17) \lim_{x \rightarrow 0} \left[\frac{\log 100 + \log(0.01+x)}{x} \right]$$

$$18) \lim_{x \rightarrow 0} \left[\frac{\log(4-x) - \log(4+x)}{x} \right]$$

19) Evaluate the limit of the function if exist at

$$x = 1 \text{ where } f(x) = \begin{cases} 7-4x & x < 1 \\ x^2 + 2 & x \geq 1 \end{cases}$$

Activity 7.1

Evaluate : $\lim_{x \rightarrow 0} \left[\frac{e^x - x - 1}{x} \right]$

Solution : $= \lim_{x \rightarrow 0} \left[\frac{(e^x - 1) - \square}{x} \right]$

$$= \lim_{x \rightarrow 0} \frac{\square}{x} - \frac{x}{x}$$

$$= \lim_{x \rightarrow 0} \left[\frac{e^x - 1}{x} \right] - \square$$

$$= \square - 1$$

$$= \square$$

Activity 7.2

Carry out the following activity.

Evaluate $\lim_{x \rightarrow 1} \left[\frac{1}{x-1} + \frac{2}{1-x^2} \right]$

Solution : $\lim_{x \rightarrow 1} \left[\frac{1}{x-1} + \frac{2}{1-x^2} \right]$

$$= \lim_{x \rightarrow 1} \left[\frac{1}{x-1} - \frac{2}{\square} \right]$$

$$= \lim_{x \rightarrow 1} \left[\frac{\square}{(x-1)(x+1)} \right]$$

$$= \lim_{x \rightarrow 1} \left[\frac{\square}{(x-1)(x+1)} \right]$$

$$= \lim_{x \rightarrow 1} \frac{1}{x+1}$$

$$= \square$$

