2 SEQUENCES AND SERIES

Let's Study

- A.P. and G.P.
- Sum of n terms of a G.P.
- Sum of infinite terms of a G.P.
- H.P. and A.G.P.
- A.M., G.M, H.M.

Let's Recall

2.1 Sequence :

A set of numbers where the numbers are arranged in a definite order,like the natural numbers, is called a sequence.

Examples : Natural numbers, Even integers between 10 and 100, Squares of integers.

In general, a sequence is written as $t_1, t_2, t_3, t_4 ..., t_n ...$ where t_1 - first term, t_4 - fourth term, ..., t_n - n^{th} term.

Finite sequence – A sequence containing finite number of terms is called a finite sequence.

It is written as $\{t_1, t_2, t_3 ..., t_n\}$ for some positive integer n.

Infinite sequence – A sequence is said to be infinite if it is not a finite sequence.

It is written as $\{t_1, t_2, t_3 ...\}$ or $\{t_n\}$ $n \geq 1$

Sequences that follow specific patterns are called **progressions.**

In the previous class, we have studied Arithmetic Progression (A.P.).

2.2 Arithmetic Progression– (A.P.)

In a sequence if the difference between any term and its preceding term $(t_{n+1} - t_n)$ is constant,then the sequence is called an Arithmetic Progression (A.P.)

Consider the following sequences

1) 2, 5, 8, 11, 14, …
2) 4, 10, 16, 22, 28, …
3) 4, 16, 64, 256, …
4) $\frac{1}{5}, \frac{1}{25}, \frac{1}{125}, \ldots$
5) −3, 2, 7, 12, 17, …

The sequences 1), 2) and 5) are A. P. but the terms in sequences 3) and 4) are not in A. P. as the difference between their consecutive terms is not constant.

If $t_1, t_2, t_3 ....., t_n$ are in A.P. then $t_{n+1} - t_n = d$, is constant for all n.

Hence the sequence can also be written as a, a+d, a+2d, ... Its n^{th} term is $t_n = a + (n-1)d$, $t_1 = a$ and sum of n terms i.e.

$$S_n = t_1 + t_2 + ... + t_n = \frac{n}{2}[2a + (n-1)d].$$

If $t_1, t_2, t_3 ....., t_n$ are in A.P. then

i) $t_1 + k$, $t_2 + k$, $t_3 + k$, ... $t_n + k$ are also in A.P.

ii) $k t_1, kt_2, kt_3, ... , kt_n$ are also in A.P. $(k \neq 0)$

Let's Learn

2.3 Geometric progression :

A sequence $t_1, t_2, t_3 ....., t_n$ ..... is G.P. (Geometric progression) if the common ratio $\frac{t_{n+1}}{t_n} = r$ is constant for all n.

Hence a G.P. can also be written as a, ar^2, ar^3, ... where a is first term and r is common ratio.

Examples :

i) 2, 4, 8, 16, ... [here $a = 2$, r = 2]

ii) $1, \frac{1}{3}, \frac{1}{9}, \frac{1}{27}, ...$ [$a = 1$, $r = \frac{1}{3}$]

iii) 1, –1, 1, –1, 1, –1, ... [$a = 1$, r = –1]

2.3.1 The General term or the n^th^ term of a G.P.

For G.P., $t_1 = a$, $t_2 = ar$, $t_3 = ar^2$, $t_4 = ar^3$, ...

If a and r are the first term and common ratio of a G.P. respectively. Then its n^th^ term is given by $t_n = ar^{n-1}$. (Verify)

Ex. Find n^th^ term of the following G.P.

i) 3, –6, 12, –24, …

Here a = 3 , r = –2

Since $t_n = ar^{n-1} = 3\,(-2)^{n-1}$

ii) $5, 1, \frac{1}{5}, \frac{1}{25}, \ldots$

Here a = 5 , $r = \frac{1}{5}$

Since $t_n = ar^{n-1} = 5\left(\frac{1}{5}\right)^{n-1} = \left(\frac{1}{5}\right)^{n-2}$.

Properties of Geometric Progression.

If $t_1, t_2, t_3, ...t_n$ are in G.P. then

(i) $\frac{1}{t_1}, \frac{1}{t_2}, \frac{1}{t_3}, ... \frac{1}{t_n}$ are also in G.P.

ii) ($k \neq 0$) $k\,t_1, kt_2, kt_3, ... kt_n$ are also in G.P.

iii) $t_1^n, t_2^n, t_3^n, ...$ are also in G.P.

SOLVED EXAMPLES

Ex. 1) Verify whether $1, \frac{-3}{2}, \frac{9}{4}, \frac{-27}{4}, \ldots$ is a G.P., if it is a G.P. Find its ninth term.

Solution : Here $t_1 = 1$, $t_2 = \frac{-3}{2}$, $t_3 = \frac{9}{4}$,

$$\text{Consider } = \frac{t_2}{t_1} = \frac{\frac{-3}{2}}{1} = \frac{-3}{2}$$

$$\text{and } \quad \frac{t_3}{t_2} = \left(\frac{9}{4}\right)\left(-\frac{2}{3}\right) = \frac{-3}{2}$$

Here the ratio of any two consecutive terms is constant hence the given sequence is a G.P.

Now $t_9 = ar^{9-1} = a\,r^8 = 1\left(\frac{-3}{2}\right)^8$

Ex. 2) Which term of the sequence

$\sqrt{3}$, 3, $3\sqrt{3}$, ... is 243?

Solution : Here $a = \sqrt{3}$, $r = \sqrt{3}$, $t_n = 243$

and $t_n = a\,r^{n-1}$

$\therefore\ a\,r^{n-1} = 243$

$\sqrt{3}\,.(\sqrt{3}\,)^{n-1} = 243 = 3^5$

$\therefore\ (\sqrt{3}\,)^n = 3^{\frac{n}{2}} = 3^5$

$\therefore\ \frac{n}{2} = 5$

$\therefore$ n = 10

$\therefore$ Tenth term of the G.P. is 243.

Ex. 3) For a G.P. If a = 3 and $t_7 = 192$ find r and t_{11}.

Solution : Given a = □, $t_7 = ar^{□} = 192$

$\therefore$ □ (r)^□^ = 192 , $r^{□} = \frac{192}{3} = 64$

$\therefore$ $r^{□} = 2^{□}$,

$\therefore$ r= 2 .

also $t_{11} = a\,r^{□} = 3\,(2)^{□} = □$.

Ex 4) In a G.P. ,if the third term is $\frac{1}{5}$ and sixth term is $\frac{1}{625}$, find its n^th^ term .

Solution : Here $t_3 = \frac{1}{5}$, $t_6 = \frac{1}{625}$

$t_3 = ar^2 = \frac{1}{5}$ ...(1),

$t_6 = ar^5 = \frac{1}{625}$ ...(2)

Divide equation (2) by equation (1), we get

$$\frac{ar^5}{ar^2} = \frac{(1/625)}{(1/5)}$$

$$r^3 = \frac{1}{125} = \frac{1}{5^3}$$

$$\therefore\ r = \frac{1}{5}.$$

Substitute in equation (2), we get

$$a\left(\frac{1}{5}\right)^2 = \frac{1}{5}$$

$$\therefore\ a = 5.$$

$$t_n = ar^{n-1} = 5\left(\frac{1}{5}\right)^{n-1} = 5 \times (5)^{1-n} = (5)^{2-n}.$$

Ex 5) If for a sequence $\{t_n\}$, $t_n = \frac{5^{n-2}}{4^{n-3}}$ show that the sequence is a G.P. Find its first term and the common ratio.

Solution: $t_n = \frac{5^{n-2}}{4^{n-3}}$

$$\therefore\ t_{n+1} = \frac{5^{n-1}}{4^{n-2}}$$

Consider $\frac{t_n+1}{t_n} = \frac{\frac{5^{n-1}}{4^{n-2}}}{\frac{5^{n-2}}{4^{n-3}}} = \frac{5^{n-1}}{4^{n-2}} \times \frac{4^{n-3}}{5^{n-2}}$

$$\frac{5^{n-1-n+2}}{4^{n-2-n+3}} = \frac{5}{4} = \text{constant},\ \forall\ n \in N.$$

The given sequence is a G.P. with $r = \frac{5}{4}$.

and $t_1 = \frac{16}{5}$.

Let's Note :

i) Three numbers in G.P. can be conveniently assumed as $\frac{a}{r}$, a, ar

iii) Assume 4 numbers in a G.P. as

$\frac{a}{r^3}, \frac{a}{r}, ar, ar^3$ (Here ratio is r^2)

iv) Assume 5 numbers in a G.P. as

$\frac{a}{r^2}, \frac{a}{r}, a, ar, ar^2$

Ex 6) Find three numbers in G.P. such that their sum is 42 and their product is 1728.

Solution : Let the three numbers in G.P. be $\frac{a}{r}, a, ar$.

As their product is 1728

$$\frac{a}{r}.a.ar = 1728$$

$$\therefore\ a^3 = 1728 = (12)^3$$

$$\therefore\ a = 12.$$

According to first condition, their sum is 42

$$\therefore\ \frac{a}{r} + a + ar = 42$$

$$\therefore\ \frac{12}{r} + 12 + 12r = 42$$

$$\therefore\ \frac{12}{r} + 12r = 30$$

Multiply by r

$$\therefore 12 + 12r^2 = 30r$$

$$\therefore 12r^2 - 30r + 12 = 0$$

Dividing by 6, we get

$$\therefore\ 2r^2 - 5r + 2 = 0$$

$$\therefore\ (2r-1)(r-2) = 0$$

$$\therefore\ 2r = 1 \text{ or } r = 2$$

$$\therefore\ r = \frac{1}{2}\quad r = 2$$

If a=12, $r = \frac{1}{2}$ then the required numbers are 24, 12, 6.

If a = 12, r =2 then the required numbers are 6, 12, 24.

$\therefore$ 24, 12, 6 or 6, 12, 24 are the three required numbers in G.P.

Ex 7) Find four numbers in G. P. such that their product is 64 and sum of the second and third number is 6.

Solution : Let the four numbers be $\frac{a}{r^3}, \frac{a}{r}$, ar , ar^3 (common ratio is r^2)

According to the first condition

$$\frac{a}{r^3} \times \frac{a}{r} \times ar \times ar^3 = 64$$

$$\therefore a^4 = 64$$

$$\therefore a = (2^6)^{1/4} = 2^{3/2}$$

$$\therefore a = 2\sqrt{2} .$$

Now using second condition $\frac{a}{r} + ar = 6$

$$\frac{2\sqrt{2}}{r} + 2\sqrt{2}r = 6.$$

Multiplying by r,

$$2\sqrt{2} + 2\sqrt{2}\ r^2 = 6r$$

Dividing by 2

$$\sqrt{2} + \sqrt{2}\ r^2 = 3r$$

$$\sqrt{2}\, r^2 - 3r + \sqrt{2} = 0 ,$$

$$\sqrt{2}\ r^2 - 2r - r + \sqrt{2} = 0,$$

$$\sqrt{2}\ r\,(r - \sqrt{2}) - 1\,(r - \sqrt{2}) = 0.$$

$$(r - \sqrt{2})(\sqrt{2}\, r - 1) = 0.$$

$$r = \sqrt{2} \text{ or } r = \frac{1}{\sqrt{2}} .$$

If $a = 2\sqrt{2}$, $r = \sqrt{2}$ then 1, 2, 4, 8 are the four required numbers in G.P.

If $a = 2\sqrt{2}$, $r = \frac{1}{\sqrt{2}}$ then 8, 4, 2, 1 are the four required numbers in G.P.

Ex 8) If p, q, r, s are in G.P. then show that

$$(q-r)^2 + (r-p)^2 + (s-q)^2 = (p-s)^2$$

Solution : Since, p, q, r, s are in G.P. $\frac{q}{p} = \frac{r}{q} = \frac{s}{r}$

$$\therefore q^2 = pr ,\ r^2 = qs ,\ qr = ps$$

consider L.H.S. $= (q - r)^2 + (r - p)^2 + (s - q)^2$

$= q^2 - 2qr + r^2 + r^2 - 2rp + p^2 + s^2 - 2sq + q^2$

$= pr - 2qr + qs + qs - 2rp + p^2 + s^2 - 2sq + pr$

$= -2qr + p^2 + s^2 = -2ps + p^2 + s^2$ ($\because qr = ps$)

$= (p-s)^2 =$ R.H.S.

Ex 9) Shraddha deposited Rs. 8000 in a bank which pays annual interest rate of 8%.She kept it with the bank for 10 years with compound interest. Find the total amount she will receive after 10 years. [given $(1.08)^{10} = 2.1575$)]

Solution:

The Amount deposited in a bank is Rs 8000 with 8% compound interest.

Each year , the ratio of the amount to the principal to that year is constant $= \frac{108}{100}$

Hence we get a G.P. of successive amounts.

We consider the amount at the end of each year for Rs 100 , the amount is 108.

The ratio of $\frac{\text{amount}}{\text{principal}} = \frac{108}{100}$.

For P = 8000 ,

the amount after 1 year is

$$8000 + 8000 \times \frac{8}{100} = 8000\left(1 + \frac{8}{100}\right)$$

$$= 8000 \times \frac{108}{100}$$

the amount after 2 years is $8000 \times \frac{108}{100} \times \frac{108}{100}$

$$= 8000 \times \left(\frac{108}{100}\right)^2$$

the amount after 3 years is $8000 \times \frac{108}{100} \times \frac{108}{100} \times \frac{108}{100}$

$$= 8000 \times \left(\frac{108}{100}\right)^3.$$

Therefore after 10 years the amount is

$$= 8000\left(\frac{108}{100}\right)^{10}$$

$$= 8000\ (1.08)^{10}$$

$$= 8000 \times 2.1575 = 17260.$$

Thus Shraddha will get Rs 17260 after 10 years.

The formula to find amount by compound interest is

$$A = P\left(1+\frac{R}{100}\right)^N$$

$$\therefore \quad \frac{A}{P} = \left(1+\frac{R}{100}\right)^N$$

Note that, ($\left(1+\frac{R}{100}\right)^N$ is a G.P.]

EXERCISE 2.1

1) Check whether the following sequences are G.P. If so, write t_n.
   i) 2, 6, 18, 54, …
   ii) 1, –5, 25, –125 …
   iii) $\sqrt{5}, \frac{1}{\sqrt{5}}, \frac{1}{5\sqrt{5}}, \frac{1}{25\sqrt{5}}, \ldots$
   iv) 3, 4, 5, 6, …
   v) 7, 14, 21, 28, …

2) For the G.P.
   i) If $r = \frac{1}{3}$, $a = 9$ find t_7
   ii) If $a = \frac{7}{243}$, $r = 3$ find t_6.
   iii) If $r = -3$ and $t_6 = 1701$, find a.
   iv) If $a = \frac{2}{3}$, $t_6 = 162$, find r .

3) Which term of the G.P. 5, 25, 125, 625, … is 5^{10}?

4) For what values of *x, the terms* $\frac{4}{3}, x, \frac{4}{27}$ are in G.P?

5) If for a sequence, $t_n = \frac{5^{n-3}}{2^{n-3}}$, show that the sequence is a G.P. Find its first term and the common ratio.

6) Find three numbers in G.P. such that their sum is 21 and sum of their squares is 189.

7) Find four numbers in G.P. such that sum of the middle two numbers is 10/3 and their product is 1.

8) Find five numbers in G. P. such that their product is 1024 and fifth term is square of the third term.

9) The fifth term of a G.P. is x , eighth term of a G.P. is y and eleventh term of a G.P. is z verify whether $y^2 = xz$.

10) If p,q,r,s are in G.P. show that p+q , q+r , r+s are also in G.P.

11) The number of bacteria in a culture doubles every hour. If there were 50 bacteria originally in the culture, how many bacteria will be there at the end of 5^{th} hour ?

12) A ball is dropped from a height 0f 80 ft. The ball is such that it rebounds $\left(\frac{3}{4}\right)^{th}$ of the height it has fallen. How high does the ball rebound on 6^{th} bounce? How high does the ball rebound on n^{th} bounce?

13) The numbers 3, x and $x + 6$ form are in G.P. Find (i) x, (ii) 20^{th} term (iii) n^{th} term.

14) Mosquitoes are growing at a rate of 10% a year. If there were 200 mosquitoes in the begining. Write down the number of mosquitoes after (i) 3 years (ii) 10 years (iii) n years.

15) The numbers $x - 6$, $2x$ and x^2 are in G.P. Find (i) x (ii) 1^{st} term (iii) n^{th} term.

Let's Learn

2.3.2 Sum of the first n terms of a G.P. (S_n)

Consider the G.P. t_1, t_2, t_3, ... t_n .... . we write the sum of first n terms

$t_1 + t_2 + t_3 + ... + t_n$ as $\sum_{r=1}^{n} t_r = S_n$

Note : $\sum$ is the notation of summation, the sum is of all t_r $(1 \leqslant r \leqslant n)$

In $\sum_{r=1}^{n}$ the variable is r.

Theorem : If $a, ar, ar^2, ..., ar^{n-1}$ $(r \neq 1)$ is a G.P. then

$S_n = a + ar^2 + ... + ar^{n-1} = t_1 + t_2 + t_3 + ... + t_n$

$$= \sum_{r=i}^{n} t_r = \frac{a(1-r^n)}{(1-r)} ... (r \neq 1)$$

Proof : Consider $S_n = a + ar + ar^2 + ... + ar^{n-1}$

$S_n = a(1+r+r^2+r^3+ ... + r^{n-1})$ ... (1)

Multiplying both sides by r we get

$r S_n = a(r+r^2+r^3+ ... + r^n)$ ... (2)

Subtract (2) from (1) we get $S_n - S_n r = a(1-r^n)$

$\therefore$ $S_n(1-r) = a(1-r^n)$

$\therefore$ $S_n = \frac{a(1-r^n)}{1-r}$, $r \neq 1$.

Let's note :

1) If r is positive and r < 1, it is convenient to write $S_n = \frac{a(1-r^n)}{1-r}$

2) If r is positive and r > 1, it is convenient to write $S_n = \frac{a(r^n-1)}{r-1}$

3) If r = 1 then G.P. is a, a, a ...a (n times), So, $S_n = a. n$

4) $S_n - S_{n-1} = t_n$

SOLVED EXAMPLES

Ex 1) If a = 1, r = 2 find S_n for the G.P.

Solution : a = 1 , r = 2

$$S_n = a\left(\frac{r^n-1}{r-1}\right) = 1 \times \left(\frac{2^n-1}{2-1}\right) = 2^n - 1 .$$

Ex 2) For a G.P. 0.02, 0.04, 0.08, 0.016, ..., find S_n.

Solution : Here a = 0.02, r = $\frac{0.04}{0.02} = 2$

$$S_n = a\left(\frac{r^n-1}{r-1}\right) = 0.02\left(\frac{2^n-1}{2-1}\right)$$

$$= 0.02 . (2^n - 1)$$

Ex 3) For the following G.P. 3, −3, 3, −3, ..., find S_n.

Solution :

Case (i)

If n is even , n = 2k

$S_{2k} = (3-3) + (3-3) + (3-3) + (3-3) + ...$ 2k terms

= (3+3+ ... k terms) + (−3, −3, −3, ... k terms)

= 3k − 3k = 0

Case (ii)

If n is odd , n = 2k +1

$S_{2k+1} = (3 + 3 + 3 ....$ k terms)
+ (−3, −3, −3, ... k terms) + 3 = 3k − 3k + 3 = 3

Ex 4) For a G.P. if a=6 , r =2 , find S_{10}.

Solution: $S_n = a\left(\dfrac{r^n - 1}{r - 1}\right)$,

$$S_{10} = 6\left(\frac{2^{10}-1}{2-1}\right) = 6\left(\frac{1023}{1}\right) = 6\,(1023) = 6138 .$$

Ex 5) How many terms of G.P.

$2, 2^2, 2^3, 2^4, \ldots$ are needed to give the sum 2046.

Solution : Here a = 2 , r = 2 , let S_n = 2046.

$$\therefore\ 2046 = a\left(\frac{r^n-1}{r-1}\right) = 2\left(\frac{2^n-1}{2-1}\right) = 2\,(2^n-1)$$

$1023 = 2^n - 1$, $2^n = 1024 = 2^{10}$ $\therefore$ n = 10

Ex 6) If for a G.P. r=2 , S_{10}=1023 , find a .

Solution : $S_{10} = a\left(\dfrac{2^{10}-1}{2-1}\right)$

$\therefore$ 1023 = a (1023)

$\therefore$ a =1.

Ex 7) For a G.P. a = 3 , r = 2 , S_n = 765 , find n .

Solution : $S_n = 765 = 3\left(\dfrac{2^n-1}{2-1}\right) = 3\,(2^n - 1)$,

$\therefore\ 255 = 2^n - 1$,

$2^n = 256 = 2^8$,

n = 8.

Ex 8) For a G.P. if S_3= 16, S_6=144, find the first term and the common ratio of the G.P.

Solution : Given

$$S_3 = a\left(\frac{r^3-1}{r-1}\right) = 16 \qquad \ldots (1)$$

$$S_6 = a\left(\frac{r^6-1}{r-1}\right) = 144 \qquad \ldots (2)$$

Dividing (2) by (1) we get,

$$\frac{S_6}{S_3} = \frac{r^6-1}{r^3-1} = \frac{144}{16}$$

$$\frac{(r^3-1)(r^3+1)}{(r^3-1)} = 9,$$

$r^3 + 1 = 9$,

$r^3 = 8 = 2^3$,

r = 2

Substitute r = 2 in (1) We get

$$a\left(\frac{2^3-1}{2-1}\right) = 16,$$

$$a\left(\frac{8-1}{2-1}\right) = 16,$$

a (7) = 16 ,

a = 16 / 7

Ex 9) Find 5+55+555+5555+ ... upto n terms.

Solution:

Let S_n,

= 5+55+555+5555+... upto n terms.

= 5 (1+11+111+... upto n terms)

$= \dfrac{5}{9}$ (9+99+999+... upto n terms)

$= \dfrac{5}{9}$ [(10–1) + (100–1) + (1000–1) + ... to n brackets]

$= \dfrac{5}{9}$ [(10 + 100 + 1000 +... upto n terms)

– (1+1+1+ …..upto n terms)]

[(a = 10, r = 10) and (a = 1, r = 1)]

$$= \frac{5}{9}\left[10\left(\frac{10^n-1}{10-1}\right) - n\right]$$

$$= \frac{5}{9}\left[\frac{10}{9}\left(10^n - 1\right) - n\right]$$

Ex 10) Find the sum to n terms 0.3+0.33+0.333+ ... n terms

Solution :

S_n = 0.3+0.33+0.333+ upto n terms

= 3 [0.1+0.11+ 0.111+ ... n terms]

Multiply and divide by 9

$= \frac{3}{9}$ [0.9+0.99+ 0.999+ ... n terms]

$= \frac{3}{9}$ [(1–0.1)+(1–0.01)+(1–0.001)+ ... to n terms]

$= \frac{3}{9}$ [(1+1+1 ...... n terms)

– (0.1+0.01+ 0.001+ ...... n terms)]

[(a = 1, r = 1) and (a = 0.1, r = 0.1)]

$$= \frac{3}{9}\left[n - 0.1\frac{(1-0.1^n)}{(1-0.1)}\right]$$

$$= \frac{3}{9}\left[n - \frac{1}{9}(1-0.1^n)\right]$$

Ex 11) Find the n^{th} term of the sequence 0.4 , 0.44, 0.444,…

Solution :

Here t_1=0.4

t_2=0.44 = 0.4 + 0.04

t_3= 0.444 = 0.4+0.04+0.004

t_n= 0.4 + 0.04 + 0.004 + 0.0004 + ... upto n terms

here t_n is the sum of first n terms of a G.P.

with a = 0.4 and r = 0.1

$$t_n = 0.4\left(\frac{1-0.1^n}{1-0.1}\right) = \frac{0.4}{0.9}[1-0.1)^n]$$

$$= \frac{4}{9}[1-(0.1)^n].$$

Ex 12) For a sequence , if Sn = $5(4^n-1)$, find the n^{th} term, hence verify that it is a G.P., Also find r.

Solution : $S_n = 5(4^n-1)$, $S_{n-1} = 5(4^{n-1}-1)$

We know that $t_n = S_n - S_{n-1}$

$= 5(4^n-1) - 5(4^{n-1}-1)$

$= 5(4^n) - 5 - 5(4^{n-1}) + 5$

$= 5(4^n - 4^{n-1})$

$= 5(4^{n-1})(4-1)$

$= 5(4^{n-1})(3)$

$\therefore t_n = 15(4^{n-1})$

Then, $t_{n+1} = 15(4^n)$

and $\frac{t_{n+1}}{t_n} = \frac{15(4^n)}{15(4^{n-1})}$

$= 4^{n-n+1} = 4$

= constant $\forall$ n $\in$ N.

r = 4.

The sequence is a G.P. with $t_n = 15(4^{n-1})$.

Ex 13) A teacher wanted to reward a student by giving some chocolates. He gave the student two choices. He could either have 60 chocolates at once or he could get 1 chocolate on the first day, 2 on the second day, 4 on the third day and so on for 6 days.Which option should the student choose to get more chocolates?

Ans : We need to find sum of chocolates in 6 days.

According to second option teacher gives 1 chocolate on the first day , 2 on the second day, 4 on the third day, and so on.Hence it is a G. P . with a = 1, r = 2 .

By using $S_n = a\left(\frac{r^n-1}{r-1}\right)$

$$S_6 = 1\left(\frac{2^6-1}{2-1}\right)$$

$$= 64 - 1 = 63$$

Hence the student should choose the second way to get more chocolates.

Ex 14) Mr. Pritesh got a job with an annual salary package of Rs. 4,00,000 with 10% annual increment. Find his salary in the 5^{th} year and also find his total earnings through salary in 10 years.

[Given $(1.1)^4 = 1.4641$, $(1.1)^{10} = 2.59374$]

Solution : In the year he will get a salary of Rs. 4,00,000.

He gets an increment of 10% so in the second year his salary will be

$$4,00,000 \times \left(\frac{110}{100}\right) = 4,40,000$$

In the third year his salary will be

$$4,00,000 \times \left(\frac{110}{100}\right)^2 \text{ and so on ...}$$

hence it is a G.P. with a = 4,00,000 & r = 1.1.

So, his salary in the fifth year will be

$$t_5 = ar^4 = 4,00,000 \times \left(\frac{110}{100}\right)^4 = 585640.$$

His total income through salary in 10 years

will be $S_{10} = a\left(\frac{r^{10}-1}{r-1}\right)$

[$\because (1.1)^{10} = 2.59374$]

$$= 4,00,000 \times \left(\frac{2.59374-1}{0.1}\right)$$

$$= 4,00,000 \left(\frac{1.59374}{0.1}\right)$$

= 4,00,000 [15.9374] = 63,74,960.

Mr. Pritesh will get Rs. 5,85,640 in the fifth year and his total earnings through salary in 10 years will be Rs 63,74,960.

EXERCISE 2.2

1) For the following G.P.s , find S_n
   - i) 3, 6, 12, 24, ...
   - ii) $p,\ q,\ \frac{q^2}{p},\ \frac{q^3}{p}, \ldots$
   - iii) 0.7, 0.07, 0.007, ...
   - iv) $\sqrt{5}, -5, 5\sqrt{5}, -25\ldots$

2) For a G.P.
   - i) $a = 2,\ r = -\frac{2}{3}$, find S_6
   - ii) If $S_5 = 1023$, r = 4, Find a

3) For a G.P.
   - i) If a = 2, r = 3, $S_n = 242$ find n.
   - ii) For a G.P. sum of first 3 terms is 125 and sum of next 3 terms is 27, find the value of r .

4) For a G.P.
   - i) If $t_3 = 20$, $t_6 = 160$, find S_7
   - ii) If $t_4 = 16$, $t_9 = 512$, find S_{10}

5) Find the sum to n terms
   - i) 3 + 33 + 333 + 3333 + ...
   - ii) 8 + 88 + 888 + 8888 + ...

6) Find the sum to n terms
   - i) 0.4 + 0.44 + 0.444 + ...
   - ii) 0.7 + 0.77 + 0.777 + ...

7) Find the sum to n terms of the sequence
   - i) 0.5, 0.05, 0.005, ...
   - ii) 0.2, 0.02, 0.002, ...

8) For a sequence, if $S_n = 2(3^n - 1)$, find the n^{th} term, hence show that the sequence is a G.P.

9) If S,P,R are the sum , product and sum of the reciprocals of n terms of a G. P. respectively, then verify that $\left[\frac{S}{R}\right]^n = P^2$.

10) If S_n, S_{2n}, S_{3n} are the sum of n,2n,3n terms of a G.P. respectively , then verify that $S_n(S_{3n} - S_{2n}) = (S_{2n} - S_n)^2$.

11) Find (i) $\sum_{r=1}^{10}(3\times 2^r)$ (ii) $\sum_{r=1}^{10} 5\times 3^r$

12) The value of a house appreciates 5% per year. How much is the house worth after 6 years if its current worth is Rs. 15 Lac. [Given : $(1.05)^5 = 1.28$, $(1.05)^6 = 1.34$]

13) If one invests Rs. 10,000 in a bank at a rate of interest 8% per annum, how long does it take to double the money by compound interest? $[(1.08)^5 = 1.47]$

2.4 Sum of infinite terms of G. P.

Consider a G.P. of the positive terms. The sum of first n terms is $\frac{a(r^n - 1)}{(r-1)} = \frac{a(1-r^n)}{(1-r)}$

If $|r| > 1$, $r - 1$ is constant but r^n approaches ∞ as n approaches ∞, so the infinite terms cannot be summed up.

If $|r| < 1$, r^n approaches 0, as n approaches ∞ and the sum Sn = $\frac{a(1-r^n)}{(1-r)}$ approaches $\frac{a}{1-r}$

Here the infinite sum $\sum_{r=1}^{\infty} t_r$ is said to be $\frac{a}{1-r}$

Example : $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots$

Solution :

here $a = 1$, $r = \frac{1}{2}$, i.e. $|r| < 1$

$\therefore$ Sum to infinity is given by

$$\frac{a}{1-r} = \frac{1}{1-\left(\frac{1}{2}\right)} = \frac{1}{\left(\frac{1}{2}\right)} = 2$$

Visual proof : This can be visualised taking a rectangle of 2×1 containing smaller rectangles of area 1, $\frac{1}{2}$, $\frac{1}{4}$, ... square units.

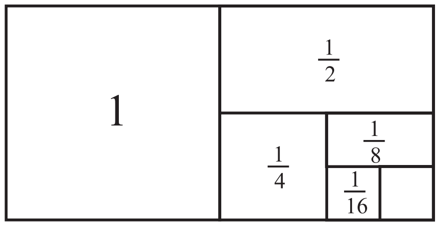

Fig. 2.1

These rectangles are seen to be completely in the big rectangle and slowly fills up the big rectangle of area two square units.

That is $1 + \frac{1}{2} + \frac{1}{4} + \ldots = 2$.

SOLVED EXAMPLES

Ex. 1) Determine whether the sum of all the terms in the series is finite. In case it is finite find it.

i) $\frac{1}{3}, \frac{1}{3^2}, \frac{1}{3^3}, \ldots$

ii) $\frac{3}{5}, \frac{-9}{25}, \frac{27}{125}, \frac{-81}{625} \ldots$

iii) $1, -3, 9, -27, 81, \ldots$

Solution:

i) Here $a = \frac{1}{3}$, $|r| = \left|\frac{1}{3}\right| = \frac{1}{3} < 1$

$\therefore$ Sum to infinite terms is finite.

$$S = \frac{a}{1-r} = \frac{\left(\frac{1}{3}\right)}{1-\left(\frac{1}{3}\right)} = \frac{\left(\frac{1}{3}\right)}{\left(\frac{2}{3}\right)} = \frac{1}{2}$$

ii) Here $a = \frac{3}{5}$, $r = -\frac{3}{5}$, $|r| = \left|\frac{-3}{5}\right| = \frac{3}{5} < 1$

$\therefore$ Sum to infinite terms is finite.

$$S = \frac{a}{1-r} = \frac{\left(\frac{3}{5}\right)}{1-(-\frac{3}{5})} = \frac{\left(\frac{3}{5}\right)}{\left(\frac{8}{5}\right)} = \frac{3}{8}$$

iii) Here a=1 , r = –3 $|r| = |-3| = 3 \not< 1$

$\therefore$ sum to infinity does not exist

2.4.1 Expressing recurring decimals as rational numbers :

We know that recurring decimal fraction can be written as rational numbers e.g. 0.666 ...$0.\overline{6}$ $= \frac{2}{3}$. This can also be checked using G.P.

Ex i) 0.66666…

= 0.6 + 0.06 + 0.006 +…

the terms are in G.P. with a=0.6, $|r| = |0.1| < 1$

$\therefore$ Sum to infinite terms is finite and is

$$\frac{a}{1-r} = \frac{0.6}{1-0.1} = \frac{0.6}{0.9} = \frac{6}{9} = \frac{2}{3}$$

ii) $0.\overline{46}$ = 0.46+0.0046+0.000046 + … the terms are in G.P. with a = 0.46 , $|r| = |0.01| < 1$.

$\therefore$ Sum to infinite terms is finite and is

$$\frac{a}{1-r} = \frac{0.46}{1-(0.01)} = \frac{0.46}{0.99} = \frac{46}{99}\ .$$

iii) $2.\overline{5}$ = 2 + 0.5 + 0. 05+ 0.005 + 0.0005 + ...

After the first term,the terms are in G.P. with a = 0.5 , $|r| = |0.1| < 1$

$\therefore$ Sum to infinite terms is finite and is

$$\frac{a}{1-r} = \frac{0.5}{1-0.1} = \frac{0.5}{0.9} = \frac{5}{9}$$

$$2.\overline{7} = 2 + \frac{5}{9} = \frac{23}{9}$$

EXERCISE 2.3

1) Determine whether the sum to infinity of the following G.P.s exist, if exists find them

i) $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots$

ii) $2, \frac{4}{3}, \frac{8}{9}, \frac{16}{27}, \ldots$

iii) $-3, 1, \frac{-1}{3}, \frac{1}{9}, \ldots$

iv) $\frac{1}{5}, \frac{-2}{5}, \frac{4}{5}, \frac{-8}{5}, \frac{16}{5}, \ldots$

v) 9, 8.1, 7.29, ...

2) Express the following recurring decimals as a rational number.

i) $0.\overline{7}$

ii) $2.\overline{4}$

iii) $2.3\overline{5}$

iv) $51.0\overline{2}$

3) If the common ratio of a G.P. is $\frac{2}{3}$ and sum to infinity is 12. Find the first term.

4) If the first term of the G.P. is 16 and its sum to infinity is $\frac{96}{17}$ find the common ratio.

5) The sum of an infinite G.P. is 5 and the sum of the squares of these terms is 15 find the G.P.

6) Find (i) $\sum_{r=1}^{\infty} 4(0.5)^r$ (ii) $\sum_{r=1}^{\infty}\left(-\frac{1}{3}\right)^r$

(iii) $\sum_{r=0}^{\infty}(-8)\left(-\frac{1}{2}\right)^n$ (iv) $\sum_{n=1}^{\infty} 0.4^n$

7) The mid points of the sides of a square of side 1 are joined to form a new square. This procedure is repeated indefinitely. Find the sum of (i) the areas of all the squares (ii) the perimeters of all the square.

8) A ball is dropped from a height of 10m. It bounces to a height of 6m, then 3.6m and so on. Find the total distance travelled by the ball.

2.5 Harmonic Progression (H. P.)

Definition : A sequence $t_1, t_2, t_3, t_4, \ldots, t_n, \ldots$ ($t_n \neq 0$, $n \in N$) is called a harmonic progression if $\frac{1}{t_1}, \frac{1}{t_2}, \frac{1}{t_3} \ldots, \frac{1}{t_n}, \ldots$ are in A.P.

e.g., i) $\frac{1}{7}, \frac{1}{11}, \frac{1}{15} \ldots$ are in H. P. as

$\frac{1}{\left(\frac{1}{7}\right)}, \frac{1}{\left(\frac{1}{11}\right)}, \frac{1}{\left(\frac{1}{15}\right)}$ i.e. 7, 11, 15 … are in A.P.

ii) $\frac{1}{4}, \frac{3}{14}, \frac{3}{16} \ldots$ is H.P. as 4, $\frac{14}{3}, \frac{16}{3}, \ldots$ are in A.P.

SOLVED EXAMPLES

Ex. 1) Find the n^{th} term of the H.P.

$\frac{1}{2}, \frac{2}{5}, \frac{1}{3}, \frac{2}{7} \ldots$

Solution : Here 2, $\frac{5}{2}$, 3, $\frac{7}{2}$ … are in A.P. with a= 2 and d = $\frac{1}{2}$ hence $\frac{1}{2}, \frac{2}{5}, \frac{1}{3}, \frac{2}{7}, \ldots$ are in H.P.

For A.P. $t_n = a + (n-1)\,d = 2 + (n-1)\,\frac{1}{2}$

$= 2 + \frac{1}{2}n - \frac{1}{2}$

$= \frac{3}{2} + \frac{n}{2}$

$= \frac{3+n}{2}$.

Therefore in H.P. $t_n = \frac{2}{3+n}$

Ex. 2) Find the n^{th} term of H.P. $\frac{1}{5}, 1, \frac{-1}{3}, \frac{-1}{7}, \ldots$

Solution: Since 5, 1, −3, −7, … are in A.P.

with a= 5 and d = −4

Hence $t_n = a + (n-1)\,d$

$= 5 + (n-1)(-4)$

$= 5 - 4n + 4 = 9 - 4n$.

For H.P. $t_n = \frac{1}{9-4n}$

2.6 Types of Means:

2.6.1 Arithmetic mean (A. M.):

If x and y are two numbers, their A.M. is given by $A = \frac{x+y}{2}$.

We observe that x, A , y form an A.P.

i.e. $A - x = y - A \quad \therefore 2A = x + y \quad \therefore A = \frac{x+y}{2}$

2.6.2 Geometric mean (G. M.) :

If x and y are two numbers having same sign (positive or negative), their G.M. is given by $G = \sqrt{xy}$.

We observe that x, G, y form a G.P.

i.e. $\frac{G}{x} = \frac{y}{G} \quad \therefore G^2 = x\,y$

$\therefore G = \sqrt{xy}$

2.6.3 Harmonic mean (H.M.) :

If x and y are two numbers, their H.M. is given by $H = \frac{2xy}{x+y}$.

We observe that x, H, y form an HP .

i.e. $\frac{1}{x}, \frac{1}{H}, \frac{1}{y}$ is in A.P.

$$\therefore \frac{1}{H} = \frac{\frac{1}{x}+\frac{1}{y}}{2} = \frac{x+y}{2xy} \quad \therefore H = \frac{2xy}{x+y}$$

Note 1) : These results can be extended to n numbers as follows

$$A = \frac{x_1 + x_2 + x_3 + \ldots + x_n}{n}$$

$$G = \sqrt[n]{x_1 . x_2 . x_3 \ldots x_n}$$

$$H = \frac{n}{\frac{1}{x_1}+\frac{1}{x_2}+\frac{1}{x_3}+....+\frac{1}{x_n}}$$

2) If $x = y$ then A = G = H

Theorem : If A, G and H are A.M., G.M., H.M. of two positive numbers x and y respectively, then

i) $G^2 = AH$ ii) $A \geq G \geq H$

Proof : Let A, G and H be A.M., G.M and H.M. of two positive numbers x and y

Then

$$A = \frac{x+y}{2}, \; G = \sqrt{xy}, \; H = \frac{2xy}{x+y}$$

i) $\quad RHS = AH = \left(\frac{x+y}{2}\right)\left(\frac{2xy}{x+y}\right)$

$= xy = G^2 = L.H.S.$

ii) $\quad$ Consider $A - G = \frac{x+y}{2} - \sqrt{xy}$

$= \frac{1}{2}(x+y-2\sqrt{xy})$

$= \frac{1}{2}\left[\left(\sqrt{x}\right)^2 + \left(\sqrt{y}\right)^2 - 2\sqrt{x}\sqrt{y}\right]$

$= \frac{1}{2}\left(\sqrt{x}-\sqrt{y}\right)^2 \geqslant 0$

Since squares are always non-negative.

$\therefore A - G \geq 0 \quad \therefore A \geq G$ (I)

$\therefore \frac{A}{G} \geq 1$ (II)

Now since $G^2 = AH$

$\frac{G}{H} = \frac{A}{G} \geq 1$ (From II)

$\therefore \frac{G}{H} \geq 1 \; \therefore G \geq H$ (III)

From (I) and (III)

$A \geq G \geq H$

SOLVED EXAMPLES

Ex. 1 : Find A.M.,G.M.,H.M. of the numbers 4 and 16

Solution : Here x = 4 and y = 16

$$A = \frac{x+y}{2} = \frac{4+16}{2} = \frac{20}{2} = 10$$

$$G = \sqrt{xy} = \sqrt{4 \times 16} = \sqrt{64} = 8$$

$$H = \frac{2xy}{x+y} = \frac{2 \times 4 \times 16}{4+16} = \frac{128}{20} = \frac{32}{5}$$

Ex. 2 : Insert 4 terms between 2 and 22 so that the new sequence is in AP.

Solution: Let A_1, A_2, A_3, A_4 be the four terms between 2 and 22 so that

2, A_1, A_2, A_3, A_4, 22 are in AP with

a = 2, t_6 = 22, n = 6 .

$\therefore$ 22 = 2 + (6–1)d = 2 + 5d

20 = 5d, d = 4

A_1 = a+d = 2+4 = 6,

A_2 = a+2d = 2+8 = 10,

$A_3 = a+3d = 2+3\times4 = 2+12 = 14$

$A_4 = a+4d = 2 + 4\times4 = 2+16 = 18.$

$\therefore$ the 4 terms between 2 and 22 are 6, 10, 14, 18.

Ex: 3 Insert two numbers between $\frac{2}{9}$ and $\frac{1}{12}$ so that the resulting sequence is a HP.

Solution : Let the required numbers be $\frac{1}{H_1}$ and $\frac{1}{H_2}$

$\therefore \frac{2}{9}, \frac{1}{H_1}, \frac{1}{H_2}, \frac{1}{12}$ are in HP.

$\therefore \frac{9}{2}, H_1, H_2, 12$ are in A.P.

$t_1 = a = \frac{9}{2}, t_4 = 12 = a+3d \therefore \frac{9}{2} + 3d = 12$

$\therefore 3d = 12 - \frac{9}{2} = \frac{24-9}{2} = \frac{15}{2}$

$\therefore d = \frac{5}{2}$

$t_2 = H_1 = a+d = \frac{9}{2} + \frac{5}{2} = \frac{14}{2} = 7.$

$t_3 = H_2 = a+2d = \frac{9}{2} + 2 \times \frac{5}{2} = \frac{19}{2}$

$\frac{1}{7}$ and $\frac{2}{19}$ are to be

inserted between $\frac{2}{9}$ and $\frac{1}{12}$

Ex: 4 Insert two numbers between 1 and 27 so that the resulting sequence is a G.P.

Solution: Let the required numbers be G_1 and G_2

$\therefore$ 1, G_1, G_2, 27 are in G.P.

$\therefore t_1 = 1, t_2 = G_1, t_3 = G_2, t_4 = 27$

$\therefore a = 1, t_4 = ar^3 = 27 \quad \therefore r^3 = 27$

$\therefore r^3 = 27 = 3^3 \therefore r = 3$

$t_2 = G_1 = ar = 1 \times 3 = 3.$

$t_3 = G_2 = ar^2 = 1(3)^2 = 9$

$\therefore$ 3 and 9 are the two required numbers between 1 and 27.

Ex: 5 The A.M. of two numbers exceeds their G.M. by 2 and their H.M. by $\frac{18}{5}$. Find the Numbers.

Solution : Given A= G + 2

$\therefore G = A - 2$ .... (I)

Also $A = H + \frac{18}{5}$ $\quad \therefore H = A - \frac{18}{5}$ ..... (II)

We know that $G^2 = AH$

$(A-2)^2 = A\left(A - \frac{18}{5}\right)$ .... from (I) and (II)

$A^2 - 4A + 2^2 = A^2 - \frac{18}{5}A$

$\frac{18}{5}A - 4A = -4$

$-2A = -4 \times 5,$

$\therefore A = 10$

Also $G = A-2 = 10 - 2 = 8$

Now, let the two numbers be x and y.

As $A = \frac{x+y}{2} = 10, \; x+y = 20,$

$\therefore y = 20 - x$ ... (III)

Now $G = \sqrt{xy} = 8$

$\therefore xy = 64$ ... (IV)

$\therefore x(20 - x) = 64$ ... from (III) and (IV)

$20x - x^2 = 64$

$x^2 - 20x + 64 = 0$

$(x-16)(x-4) = 0$

$x = 16$ or $x = 4$

If $x = 16$ then $y = 4$

If $x = 4$ then $y = 16$.

The required numbers are 4 and 16.

EXERCISE 2.4

1) Verify whether the following sequences are H.P.

   i) $\frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \frac{1}{9}, \ldots$

   ii) $\frac{1}{3}, \frac{1}{6}, \frac{1}{12}, \frac{1}{24}, \ldots$

   iii) $5, \frac{10}{17}, \frac{10}{32}, \frac{10}{47}, \ldots$

2) Find the n^{th} term and hence find the 8^{th} term of the following HPs

   i) $\frac{1}{2}, \frac{1}{5}, \frac{1}{8}, \frac{1}{11}, \ldots$

   ii) $\frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \frac{1}{10}, \ldots$

   iii) $\frac{1}{5}, \frac{1}{10}, \frac{1}{15}, \frac{1}{20}, \ldots$

3) Find A.M. of two positive numbers whose G.M. and H. M. are 4 and $\frac{16}{5}$ respectively.

4) Find H.M. of two positive numbers A.M. and G.M. are $\frac{15}{2}$ and 6

5) Find GM of two positive numbers whose A.M. and H.M. are 75 and 48

6) Insert two numbers between $\frac{1}{4}$ and $\frac{1}{3}$ so that the resulting sequence is a HP.

7) Insert two numbers between 1 and -27 so that the resulting sequence is a G.P.

8) If the A.M. of two numbers exceeds their G.M. by 2 and their H.M. by $\frac{18}{5}$, find the numbers.

9) Find two numbers whose A.M. exceeds their GM by $\frac{1}{2}$ and their HM by $\frac{25}{26}$

Let's Learn

2.7 ARITHMETICO-GEOMETRIC PROGRESSION (A.G.P.) :

Definition : A sequence in which each term is the product of the corresponding terms of an A.P. and G.P. is called and arithmetico – geometric progression (AGP).

e.g. Consider the sequence

	(a)	$(a+d)$	$(a+2d)$	$(a+3d)$
A.P.	2	5	8	11
G.P.	1	3	9	27
	(1)	(r)	(r^2)	(r^3)

A.G.P. is a, $(a+d)r$, $(a+2d)\, r^2$, $(a+3d)\, r^3$, …

i.e. $2\times1, 5\times3, 8\times9, 11\times27, \ldots$

Here the first factors of the terms are in AP and the second factors are in G.P.

Therefore the given sequence forms an A.G.P.

n^{th} term of A.G.P.

$t_1 = a\times1,\ t_2 = (a+d)r,\ t_3 = (a+2d)r^2, \ldots$

$\therefore\ t_n = [a + (n-1)d]\ r^{n-1}$

2.7.1 Sum of n terms of A.G.P.:

Let $S_n = a + (a + d)r + (a + 2d)r^2 + \ldots$

$+ [a + (n-1)d]r^{n-1}$

$\therefore\ S_n = a + ar + dr + ar^2 + 2dr^2 + \ldots.$

$+ ar^{n-1} + (n-1)dr^{n-1}$ ...(i)

$\therefore\ rS_n = ar + ar^2 + dr^2 + ar^3 + 2dr^3 + \ldots$

$+ ar^n + (n-1)dr^n$ ...(ii)

Subtracting (ii) from (i) we get

$S_n - rS_n = a + dr + dr^2 + ... dr^{n-1} - ar^n - (n-1)dr^n$

$\therefore S_n(1-r) = a + dr + dr^2 + ...$

$+ dr^{n-1} - [a + n(n-1)d]r^n$

$$S_n(1-r) = a + dr\left(\frac{1-r^{n-1}}{(1-r)}\right) - [a+(n-1)d]r^n$$

$$S_n = \frac{a}{1-r} + \frac{dr(1-r^{n-1})}{(1-r)^2} - \frac{[a+(n-1)d]r^n}{1-r}$$

$(r \neq 1)$

Note that, sum to infinity of A.G.P. is

$$\frac{a}{1-r} + \frac{dr}{(1-r)^2}$$

SOLVED EXAMPLE

Ex 1) Find t_n and the sum of n terms of

1, 4, 12, 32, 80, 192, ...

Solution: Given sequence can be written as

1×1, 2×2, 3×4, 4×8, 5×16, …

We observe that first factors in each term 1, 2, 3, 4, 5, ... are in A.P. with a = 1,d = 1

and the second factors in each term 1, 2, 4, 8, 16, … are in G.P with r = 2

$\therefore t_n = [a + (n-1)d]r^{n-1}$

$= [1 + (n-1)]2^{n-1} = n.2^{n-1}$

We know that S_n of A.G.P. is given by

$$S_n = \frac{a}{1-r} + dr\frac{(1-r^{n-1})}{(1-r)^2} - \frac{[a+(n-1)d]r^n}{1-r} \quad (r \neq 1)$$

Substituting the values of a,d,r we get ,

$$S_n = \frac{1}{1-2} + 1 \times 2\frac{(1-2^{n-1})}{(1-2)^2} - \frac{[1+(n-1)1]2^n}{1-2}$$

$$S_n = \frac{1}{-1} + 2(1-2^{n-1}) + n.\,2^n$$

$$S_n = -1 + 2 - 2^n + n\,.\,2^n$$

$$S_n = 1 - 2^n + n\,2^n$$

$$S_n = 1 - 2^n(1-n) = 1 + (n-1)\,2^n\,.$$

EXERCISE 2.5

1) Find S_n of the following arithmetico – geometric sequences

i) $2, 4x, 6x^2, 8x^3, 10x^4, \ldots$

ii) $1, 4x, 7x^2, 10x^3, 13x^4, \ldots$

iii) 1, 2×3, 3×9, 4×27, 5×81, …

iv)) 3, 12, 36, 96, 240, …

2) Find the sum to infinity of the following arithmetico – geometric sequence

i) $1, \frac{2}{4}, \frac{3}{16}, \frac{4}{64}, \ldots$

ii) $3, \frac{6}{5}, \frac{9}{25}, \frac{12}{125}, \frac{15}{625}, \ldots$

iii) $1, \frac{-4}{3}, \frac{7}{9}, \frac{-10}{27} \ldots$

Properties of Summation

i) $\sum_{r=1}^{n} kt_r = k\sum_{r=1}^{n} t_r$, ... $k \neq 0$

where k is a nonzero constant.

ii) $\sum_{r=1}^{n} (a_r + b_r) = \sum_{r=1}^{n} a_r + \sum_{r=1}^{n} b_r$

iii) $\sum_{1}^{n} 1 = n$

iv) $\sum_{1}^{n} k = k\sum_{1}^{n} 1 = kn$, ... $k \neq 0$

Result: 1) The Sum of the first n natural numbers

$$= \sum_{r=1}^{n} r = \frac{n(n+1)}{2}$$

Result 2) The Sum of squares of first n natural numbers

$$= \sum_{r=1}^{n} r^2 = \frac{n(n+1)(2n+1)}{6}$$

Result 3) The sum of the cubes of the first n natural numbers $= \sum_{r=1}^{n} r^3 = \left(\frac{n(n+1)}{2}\right)^2$

The above results can be proved using Mathematical Induction. (Chapter No. 4)

SOLVED EXAMPLES

Ex 1) Evaluate $\sum_{r=1}^{n} (8r - 7)$

Solution : $\sum_{r=1}^{n} (8r - 7) = \sum_{r=1}^{n} 8r - \sum_{r=1}^{n} 7$

$= 8 \sum_{r=1}^{n} r - 7 \sum_{r=1}^{n} 1 = 8\left(\frac{n(n+1)}{2}\right) - 7n$

$= 4(n^2+n) - 7n = 4n^2 + 4n - 7n = 4n^2 - 3n.$

Ex 2) Find $3^2 + 4^2 + 5^2 + \ldots + 29^2$.

Solution: $3^2 + 4^2 + 5^2 + \ldots + 29^2$

$= (1^2 + 2^2 + 3^2 + \ldots + 29^2) - (1^2 + 2^2)$

$= \sum_{r=1}^{29} r^2 - \sum_{r=1}^{2} r^2$

$= \frac{29(29+1)(58+1)}{6} - \frac{2(2+1)(4+1)}{6}$

$= (29 \times 30 \times 59) / 6 - (2 \times 3 \times 5) / 6$

$= (29 \times 5 \times 59) - 5$

$= 5 (29 \times 59 - 1) = 5 (1711 - 1)$

$= 5 (1710) = 8550$

Ex 3) Find $100^2 - 99^2 + 98^2 - 97^2 + \ldots + 2^2 - 1^2$

Solution : $100^2 - 99^2 + 98^2 - 97^2 + \ldots + 2^2 - 1^2$

$= (100^2 + 98^2 + 96^2 + \ldots + 2^2) - (99^2 + 97^2 + 95^2 + \ldots + 1^2)$

$= \sum_{r=1}^{50} (2r)^2 - \sum_{r=1}^{50} (2r - 1)^2$

$= \sum_{r=1}^{50} (4r^2 - 4r^2 + 4r - 1)$

$= \sum_{r=1}^{50} (4r - 1)$

$= \sum_{r=1}^{50} 4r - \sum_{r=1}^{50} 1$

$= 4\sum_{r=1}^{50} r - \sum_{r=1}^{50} 1 = 4 \times \frac{50(50+1)}{2} - 50$

$= 2 \times 50 \times 51 - 50$

$= 50 (2 \times 51 - 1)$

$= 50 (101)$

$= 5050.$

Ex 4) Find $1 \times 5 + 3 \times 7 + 5 \times 9 + 7 \times 11$ ... upto n terms.

Solution : Note that first factors of each term 1, 3, 5, 7, ... are in A.P. with a =1, d=2.

$t_r = a + (r-1)d = 1 + (r-1)2 = 2r-1.$

Also the second factors 5,7,9,11, ... are also in A.P. with a= 5 , d = 2 .

$t_r = 5 + (r-1)2 = 5 + 2r - 2 = 2r+3$

$\therefore S_n = \sum_{r=1}^{n} (2r - 1)(2r + 3)$

$= \sum_{r=1}^{n} (4r^2 + 4r - 3)$

$= 4\sum_{r=1}^{n} r^2 + 4\sum_{r=1}^{n} r - \sum_{r=1}^{n} 3$

$= 4 \frac{n(n+1)(2n+1)}{6} + 4 \frac{n(n+1)}{2} - 3n$

$= n\left[\frac{2(n+1)(2n+1)}{3}\right] + 2n(n+1) - 3n$

$= \frac{n}{3} [2(2n^2 + n + 2n + 1) + 6(n + 1) - 9]$

$= \frac{n}{3} [(4n^2 + 6n + 2) + 6n - 3]$

$= \frac{n}{3} (4n^2 + 12n - 1)$

2.8 Power Series

Some functions can be expressed as infinite sums of powers of x. They are called powers series.

Some examlpes of Power Series are,

1) $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots$

2) $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$

3) $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$

4) $e^{-x} = 1 - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} \ldots$

5) If $|x| < 1$ then

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}$$

The proofs of the equations given above are obtained at more advanced stage in mathematics.

EXERCISE 2.6

1) Find the sum $\sum_{r=1}^{n} (r+1)(2r-1)$

2) Find $\sum_{r=1}^{n} (3r^2 - 2r + 1)$

3) Find $\sum_{r=1}^{n} \left(\frac{1+2+3.....+r}{r}\right)$

4) Find $\sum_{r=1}^{n} \left(\frac{1^3+2^3+.....+r^3}{r(r+1)}\right)$

5) Find the sum $5 \times 7 + 9 \times 11 + 11 \times 3 + \ldots$ upto n terms.

6) Find the sum $2^2+4^2+6^2+8^2+\ldots$ upto n terms.

7) Find $(70^2 - 69^2) + (68^2-67^2)+(66^2-65^2) + \ldots + (2^2-1^2)$

8) Find the sum $1\times3\times5 + 3\times5\times7 + 5\times7\times9 + \ldots + n(n+1)(n+4)$

9) If $\frac{1\times2 + 2\times3 + 3\times4 + 4\times5 + \ldots \text{ upto n terms}}{1+2+3+4+\ldots \text{ upto n terms}} = \frac{100}{3}$, find n.

10) If S_1, S_2 and S_3 are the sums of first n natural numbers, their squares and their cubes respectively then show that -

$9\,S_2^{\,2} = S_3\,(1+8\,S_1)$.

Let's Remember

1) For A.P. $t_n = a + (n-1)\,d$

2) For G.P. $t_n = a\,r^{n-1}$, $S_n = \frac{a(r^n-1)}{(r-1)}$

3) A.M. of two numbers $A = \frac{x+y}{2}$

4) G.M. of two numbers $G = \sqrt{xy}$

5) H.M. of two numbers $H = \frac{2xy}{x+y}$

6) $G^2 = AH$

7) If $x = y$ then $A = G = H$

8) If $x \neq y$ then $H < G < A$.

MISCELLANEOUS EXERCISE - 2

(I) Select the correct answer from the given alternative.

1) The common ratio for the G.P. 0.12,0.24,0.48, is –

A) 0.12 B) 0.2 C) 0.02 D) 2 .

2) The tenth term of the geometric sequence $\frac{1}{4}, \frac{-1}{2}, 1, -2, \ldots$ is –

A) 1024 B) $\frac{1}{1024}$ C) -128 D) $\frac{-1}{128}$

3) If for a G.P. $\frac{t_6}{t_3} = \frac{1458}{54}$ then r = ?

A) 3 B) 2 C) 1 D) −1.

4) Which term of the geometric progression 1, 2, 4, 8, ... is 2048.

A) 10^{th} B) 11^{th} C) 12^{th} D) 13^{th}

5) If common ratio of the G.P is 5 , 5^{th} term is 1875 , the first term is -

A) 3 B) 5 C) 15 D) −5

6) The sum of 3 terms of a G.P. is $\frac{21}{4}$ and their product is 1 then the common ratio is –

A) 1 B) 2 C) 4 D) 8

7) Sum to infinity of a G.P. $5, -\frac{5}{2}, \frac{5}{4}, -\frac{5}{8}, \frac{5}{16}, \ldots$ is –

A) 5 B) $-\frac{1}{2}$ C) $\frac{10}{3}$ D) $\frac{3}{10}$

8) The tenth term of H.P. $\frac{2}{9}, \frac{1}{7}, \frac{2}{19}, \frac{1}{12}, \ldots$ is -

A) $\frac{1}{27}$ B) $\frac{9}{2}$ C) $\frac{5}{2}$ D) 27.

9) Which of the following is not true, where A, G, H are the AM, GM, HM of a and b respectively. (a, b > 0)

A) $A = \frac{a+b}{2}$ B) $G = \sqrt{ab}$

C) $H = \frac{2ab}{a+b}$ D) A = GH .

10) The G.M.of two numbers exceeds their H.M. by $\frac{6}{5}$, the A.M. exceeds G.M. by $\frac{3}{2}$ the two numbers are ...

A) 6, 15/2 B) 15, 25

C) 3, 12 D) 6/5, 3/2 .

(II) Answer the following.

1) In a G.P., the fourth term is 48 and the eighth term is 768. Find the tenth term.

2) Find the sum of the first 5 terms of the G.P. whose first term is 1 and common ratio is $\frac{2}{3}$

3) For a G.P. $a = \frac{4}{3}$ and $t_7 = \frac{243}{1024}$, find the value of r .

4) For a sequence , if $t_n = \frac{5^{n-2}}{7^{n-3}}$, verify whether the sequence is a G.P. If it is a G.P. , find its first term and the common ratio.

5) Find three numbers in G.P. such that their sum is 35 and their product is 1000.

6) Find five numbers in G.P. such that their product is 243 and sum of second and fourth number is 10.

7) For a sequence $S_n = 4(7^n - 1)$ verify that the sequence is a G.P.

8) Find 2+22+222+2222+ ... upto n terms.

9) Find the n^{th} term of the sequence 0.6,0.66,0.666,0.6666, ...

10) Find $\sum_{r=1}^{n} (5r^2 + 4r - 3)$

11) Find $\sum_{r=1}^{n} r(r-3)(r-2)$

12) Find $\sum_{r=1}^{n} \frac{1^2 + 2^2 + 3^2 + \ldots + r^2}{2r+1}$

13) Find $\sum_{r=1}^{n} \frac{1^3 + 2^3 + 3^3 + \ldots r^3}{(r+1)^2}$

14) Find $2 \times 6 + 4 \times 9 + 6 \times 12 + \ldots$ upto n terms.

15) Find $2 \times 5 \times 8 + 4 \times 7 \times 10 + 6 \times 9 \times 12 + \ldots$ upto n terms.

16) Find $\frac{1^2}{1} + \frac{1^2+2^2}{2} + \frac{1^2+2^2+3^2}{3} + \ldots$ upto n terms.

17) Find $12^2 + 13^2 + 14^2 + 15^2 + \ldots 20^2$

18) If $\frac{1+2+3+4+5+\ldots \text{ upto n terms}}{1\times2+2\times3+3\times4+4\times5+\ldots \text{ upto n terms}} = \frac{3}{22}$

Find the value of n .

19) Find $(50^2 - 49^2) + (48^2 - 47^2) + (46^2 - 45^2) + \ldots + (2^2 - 1^2)$.

20) If $\frac{1\times3+2\times5+3\times7+\ldots \text{ upto n terms}}{1^3+2^3+3^3+\ldots \text{upto n terms}} = \frac{5}{9}$, find the value of n.

21) For a G.P. if $t_2 = 7$, $t_4 = 1575$ find a

22) If for a G.P. $t_3 = 1/3$, $t_6 = 1/81$ find r

23) Find $\sum_{r=1}^{n} \left(\frac{2}{3}\right)^r$.

24) Find k so that k−1,k , k+2 are consecutive terms of a G.P.

25) If for a G.P. first term is $(27)^2$ and seventh term is $(8)^2$, find S_8.

26) If p^{th}, q^{th} and r^{th} terms of a G.P. are x,y,z respectively .Find the value of $x^{q-r} . y^{r-p} . z^{p-q}$

27) Which 2 terms are inserted between 5 and 40 so that the resulting sequence is G.P.

28) If p,q,r are in G.P. and $p^{1/x} = q^{1/y} = r^{1/z}$,verify whether x,y,z are in A.P. or G.P. or neither.

29) If a,b,c are in G.P. and $ax^2+2bx+c=0$ and $px^2+2qx+r=0$ have common roots then verify that $p b^2 - 2 q b a + r a^2 = 0$

30) If p,q,r,s are in G.P., show that $(p^2+ q^2 + r^2)(q^2 +r^2+ s^2) = (pq+qr+rs)^2$

31) If p,q,r,s are in G.P. , show that $(p^n + q^n)$, $(q^n + r^n)$, (r^n+s^n) are also in G.P.

32) Find the coefficientof x^6 in the expansion of e^{2x} using series expansion .

33) Find the sum of infinite terms of

$$1 + \frac{4}{5} + \frac{7}{25} + \frac{10}{125} + \frac{13}{625} + \ldots$$