



6 FUNCTIONS



Let's Study

- Function, Domain, Co-domain, Range
- Types of functions
- Representation of function
- Basic types of functions
- Piece-wise defined and special functions



Let's : Learn

6.1 Function

Definition : A function (or mapping) f from a set A to set B ($f: A \rightarrow B$) is a relation which associates for each element x in A , a unique (exactly one) element y in B .

Then the element y is expressed as $y = f(x)$.

y is the image of x under f .

f is also called a map or transformation.

If such a function exists, then A is called the **domain** of f and B is called the **co-domain** of f .

Illustration:

Examine the following relations which are given by arrows of line segments joining elements in A and elements in B .

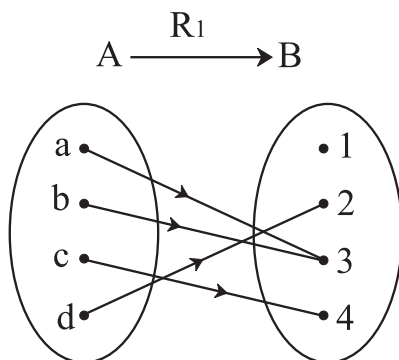


Fig. 6.1

Since, every element from A is associated to exactly one element in B , R_1 is a well defined function.

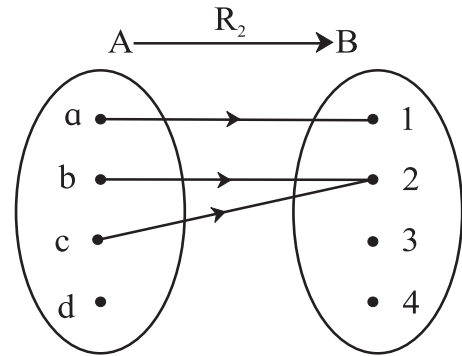


Fig. 6.2

R_2 is not a function because element 'd' in A is not associated to any element in B .

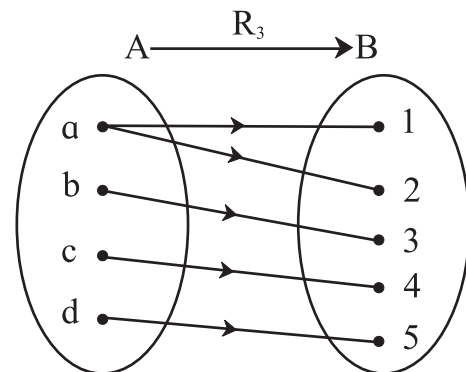


Fig. 6.3

R_3 is not a function because element a in A is associated to two elements in B .

The relation which defines a function f from domain A to co-domain B is often given by an algebraic rule.

For example, $A = \mathbb{Z}$, the set of integers and $B = \mathbb{Q}$ the set of rational numbers and the function

f is given by $f(n) = \frac{n}{7}$ here $n \in \mathbb{Z}$, $f(n) \in \mathbb{Q}$.

6.1.1 Types of function

One-one or One to one or Injective function

Definition : A function $f: A \rightarrow B$ is said to be one-one if different elements in A have different images in B. The condition is also expressed as

$$f(a) = f(b) \Rightarrow a = b \quad [\text{As } a \neq b \Rightarrow f(a) \neq f(b)]$$

Onto or Surjective function

Definition: A function $f: A \rightarrow B$ is said to be onto if every element y in B is an image of some x in A (or y in B has preimage x in A)

The image of A can be denoted by $f(A)$.

$$f(A) = \{y \in B \mid y = f(x) \text{ for some } x \in A\}$$

$f(A)$ is also called the **range** of f .

Note that $f: A \rightarrow B$ is onto if $f(A) = B$.

Also range of $f = f(A) \subset \text{co-domain of } f$.

Illustration:

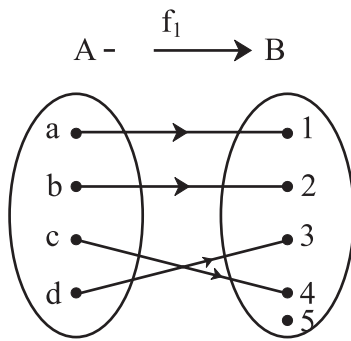


Fig. 6.4

f_1 is one-one, but not onto as element 5 is in B has no pre image in A

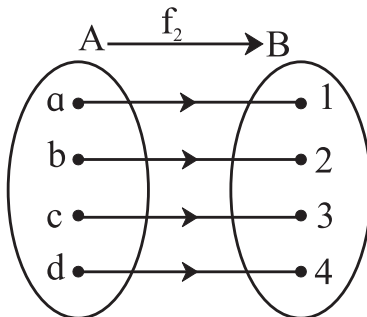


Fig. 6.5

f_2 is one-one, and onto

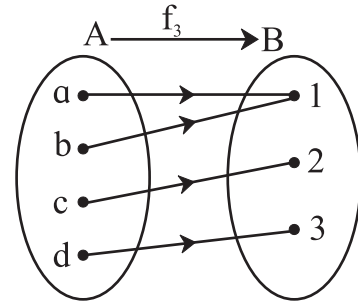


Fig. 6.6

f_3 is onto but not one-one as $f(a) = f(b) = 1$ but $a \neq b$.

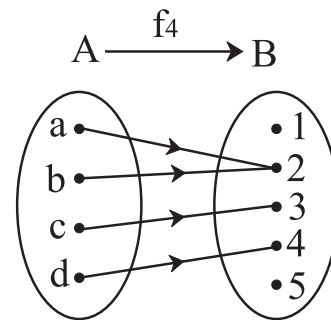
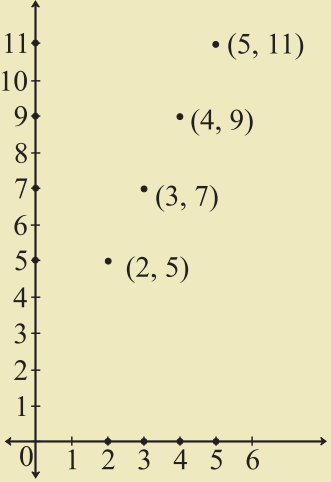


Fig. 6.7

f_4 is neither one-one, nor onto

6.1.2 Representation of Function

Verbal form	Output exceeds twice the input by 1 Domain : Set of inputs Range : Set of outputs
Arrow form on Venn Diagram	<p>Fig. 6.8</p> Domain : Set of pre-images Range: Set of images
Ordered Pair (x, y)	$f = \{(2,5), (3,7), (4,9), (5,11)\}$ Domain : Set of 1 st components from each ordered pair = $\{2, 3, 4, 5\}$ Range : Set of 2 nd components from each ordered pair = $\{5, 7, 9, 11\}$

Rule / Formula	$y = f(x) = 2x + 1$ Where $x \in N, 1 < x < 6$ $f(x)$ read as 'f of x' or 'function of x' Domain : Set of values of x for which $f(x)$ is defined Range : Set of values of y for which $f(x)$ is defined										
Tabular Form	<table border="1" style="display: inline-table; margin-right: 20px;"> <thead> <tr><th>x</th><th>y</th></tr> </thead> <tbody> <tr><td>2</td><td>5</td></tr> <tr><td>3</td><td>7</td></tr> <tr><td>4</td><td>9</td></tr> <tr><td>5</td><td>11</td></tr> </tbody> </table> Domain : x values Range: y values	x	y	2	5	3	7	4	9	5	11
x	y										
2	5										
3	7										
4	9										
5	11										
Graphical form	 <p style="text-align: center;">Fig. 6.9</p> Domain: Projection of graph on x -axis. Range: Projection of graph on y -axis.										

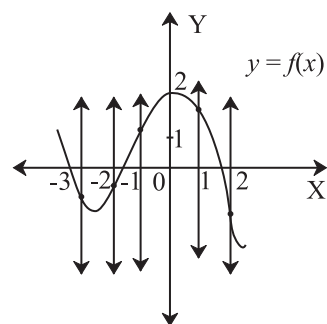


Fig. 6.10

Since every x has a unique associated value of y .
 It is a function.

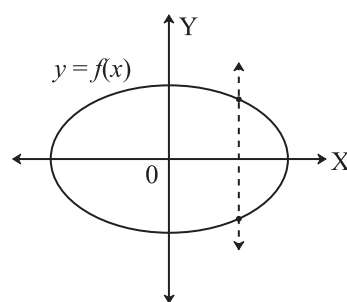


Fig. 6.11

This graph does not represent a function as vertical line intersects at more than one point some x has more than one values of y .

Horizontal Line Test:

If no horizontal line intersects the graph of a function in more than one point, then the function is one-one function.

Illustration:

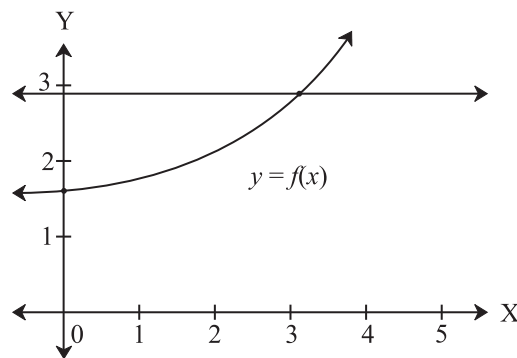


Fig. 6.12

6.1.3 Graph of a function:

If the domain of function is in R , we can show the function by a graph in xy plane. The graph consists of points (x,y) , where $y = f(x)$.

Vertical Line Test

Given a graph, let us find if the graph represents a function of x i.e. $f(x)$

A graph represents function of x , only if no vertical line intersects the curve in more than one point.

The graph is a one-one function as a horizontal line intersects the graph at only one point.

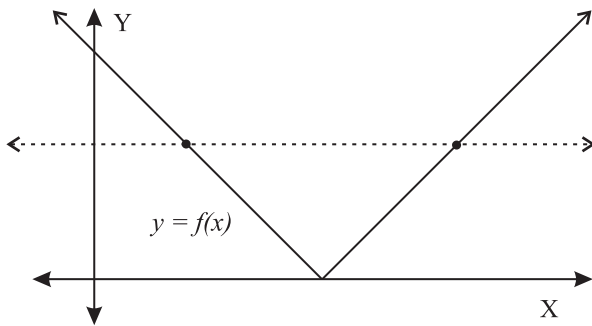


Fig. 6.13

The graph is a one-one function

6.1.4 Value of function : $f(a)$ is called the value of function $f(x)$ at $x = a$

Evaluation of function:

Ex. 1) Evaluate $f(x) = 2x^2 - 3x + 4$ at $x = 7$ & $x = -2t$

Solution : $f(x)$ at $x = 7$ is $f(7)$

$$\begin{aligned} f(7) &= 2(7)^2 - 3(7) + 4 \\ &= 2(49) - 21 + 4 \\ &= 98 - 21 + 4 \\ &= 81 \end{aligned}$$

$$\begin{aligned} f(-2t) &= 2(-2t)^2 - 3(-2t) + 4 \\ &= 2(4t^2) + 6t + 4 \\ &= 8t^2 + 6t + 4 \end{aligned}$$

Ex. 2) Using the graph of $y = g(x)$, find $g(-4)$ and $g(3)$

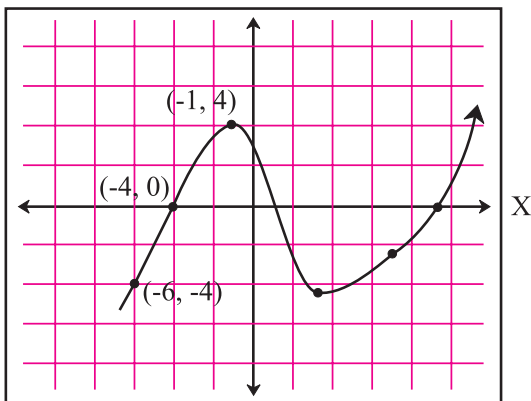


Fig. 6.14

Solution : From graph when $x = -4$, $y = 0$
so $g(-4) = 0$

From graph when $x = 3$, $y = -5$ so $g(3) = -5$

Function Solution:

Ex. 3) If $t(m) = 3m^2 - m$ and $t(m) = 4$, then find m

Solution : As

$$\begin{aligned} t(m) &= 4 \\ 3m^2 - m &= 4 \\ 3m^2 - m - 4 &= 0 \\ 3m^2 - 4m + 3m - 4 &= 0 \\ m(3m - 4) + 1(3m - 4) &= 0 \\ (3m - 4)(m + 1) &= 0 \end{aligned}$$

Therefore, $m = \frac{4}{3}$ or $m = -1$

Ex. 4) From the graph below find x for which $f(x) = 4$

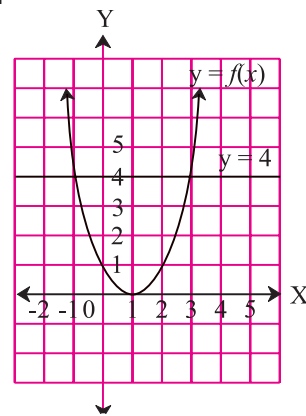


Fig. 6.15

Solution : To solve $f(x) = 4$ i.e. $y = 4$

Find the values of x where graph intersects line $y = 4$

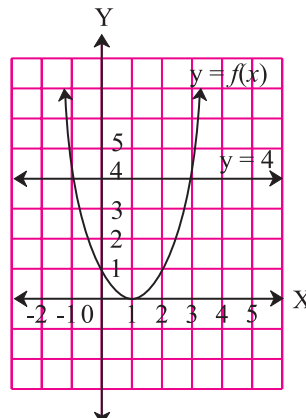


Fig. 6.16

Therefore, $x = -1$ and $x = 3$.

Function from equation:

Ex. 5) (Activity) From the equation $4x + 7y = 1$ express

- i) y as a function of x
- ii) x as a function of y

Solution : Given equation is $4x + 7y = 1$

- i) From the given equation

$$7y = \square$$

$$y = \square = \text{function of } x$$

$$\text{So } y = f(x) = \square$$

- ii) From the given equation

$$4x = \square$$

$$x = \square = \text{function of } y$$

$$\text{So } x = g(y) = \square$$

6.1.5 Some Basic Functions

(Here $f: \mathbb{R} \rightarrow \mathbb{R}$ Unless stated otherwise)

1) Constant Function

Form : $f(x) = k, k \in \mathbb{R}$

Example : Graph of $f(x) = 2$

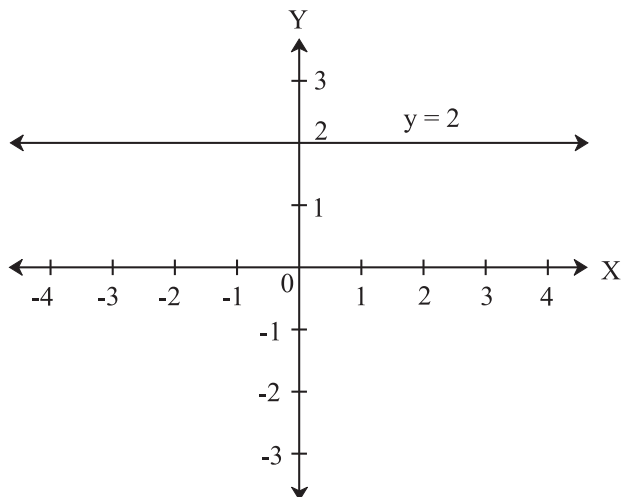


Fig. 6.17

Domain : \mathbb{R} or $(-\infty, \infty)$ and **Range :** $\{2\}$

2) Identity function

If $f: \mathbb{R} \rightarrow \mathbb{R}$ then identity function is defined by $f(x) = x$, for every $x \in \mathbb{R}$.

Identity function is given in the graph below.

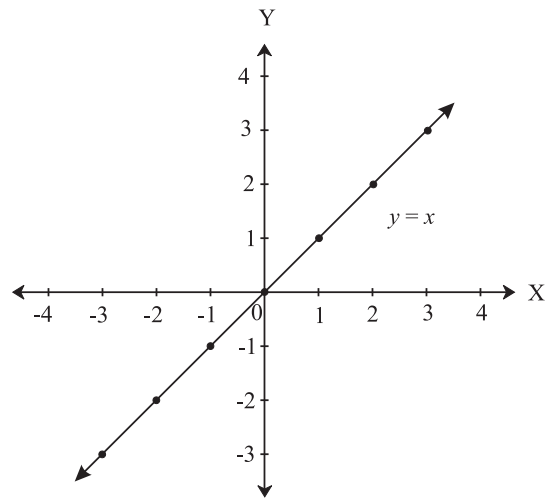


Fig. 6.18

Domain : \mathbb{R} or $(-\infty, \infty)$ and **Range :** \mathbb{R} or $(-\infty, \infty)$

[**Note :** Identity function is also given by $I(x) = x$].

3) Power Functions : $f(x) = ax^n, n \in \mathbb{N}$

(Note that this function is a multiple of n^{th} power of x)

i) Square Function

Example : $f(x) = x^2$

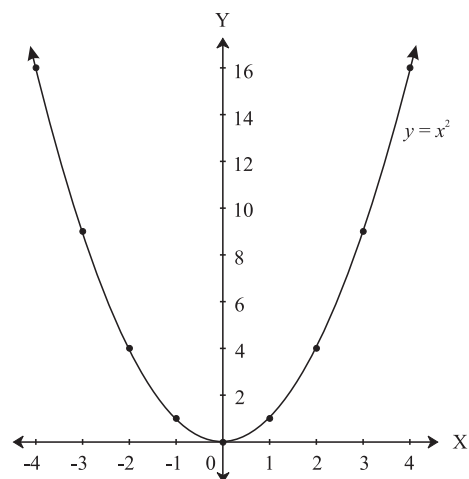


Fig. 6.19

Domain : \mathbb{R} or $(-\infty, \infty)$ and **Range :** $[0, \infty)$

Properties:

- 1) Graph of $f(x) = x^2$ is a parabola opening upwards and with vertex at origin.
- 2) Graph is symmetric about y - axis .
- 3) The graph of even powers of x looks similar to square function. (verify !) e.g. x^4, x^6 .
- 4) $(y - k) = (x - h)^2$ represents parabola with vertex at (h, k)
- 5) If $-2 \leq x \leq 2$ then $0 \leq x^2 \leq 4$ (see fig.) and if $-3 \leq x \leq 2$ then $0 \leq x^2 \leq 9$ (see fig).

ii) Cube Function

Example : $f(x) = x^3$

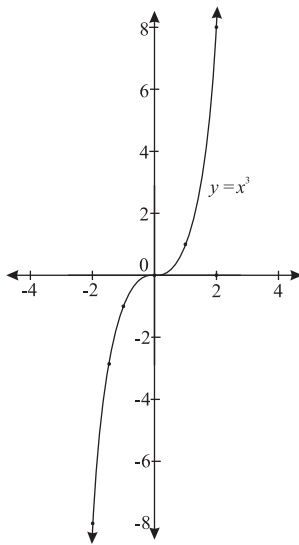


Fig. 6.20

Domain : \mathbb{R} or $(-\infty, \infty)$ and **Range :** \mathbb{R} or $(-\infty, \infty)$

Properties:

- 1) The graph of odd powers of x (more than 1) looks similar to cube function. e.g. x^5, x^7 .

4) Polynomial Function

$$f(x) = a_0 x^n + a_1 x^{n-1} + \dots + a_{n-1} x + a_n$$

is polynomial function of degree n , if $a_0 \neq 0$, and a_i s are real.

i) Linear Function

Form : $f(x) = ax + b$ ($a \neq 0$)

Example : $f(x) = -2x + 3, x \in \mathbb{R}$

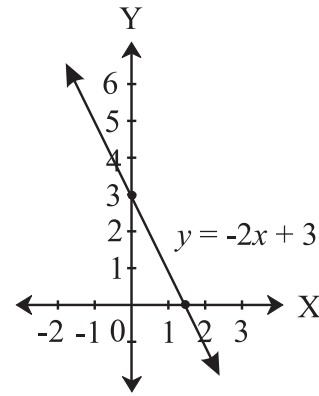


Fig. 6.21

Domain : \mathbb{R} or $(-\infty, \infty)$ and **Range :** \mathbb{R} or $(-\infty, \infty)$

Properties :

- 1) Graph of $f(x) = ax + b$ is a line with slope 'a', y -intercept 'b' and x -intercept $\left(-\frac{b}{a}\right)$.
- 2) Function : is increasing when slope is positive and decreasing when slope is negative.

ii) Quadratic Function

Form : $f(x) = ax^2 + bx + c$ ($a \neq 0$)

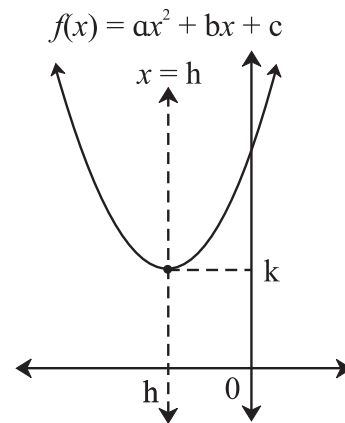


Fig. 6.22

Domain : \mathbb{R} or $(-\infty, \infty)$ and **Range :** $[k, \infty)$

Properties :

- 1) Graph of $f(x) = ax^2 + bx + c$ and where $a > 0$ is a parabola.

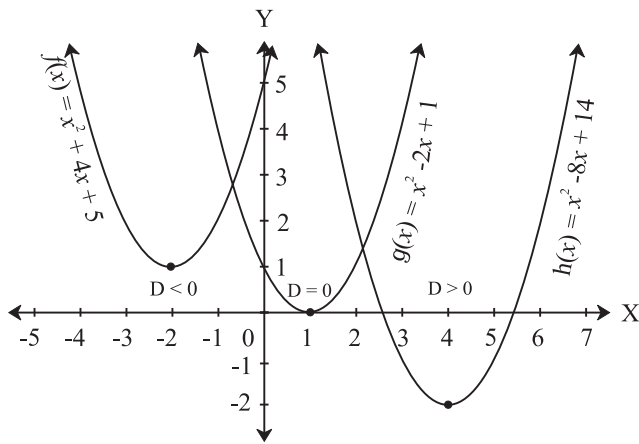


Fig. 6.23

Consider, $y = ax^2 + bx + c$

$$= a \left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} \right) + c - \frac{b^2}{4a}$$

$$= a \left(x + \frac{b}{2a} \right)^2 - \frac{b^2 - 4ac}{4a}$$

$$\left(y + \frac{b^2 - 4ac}{4a} \right) = a \left(x + \frac{b}{2a} \right)^2$$

With change of variable

$$X = x + \frac{b}{2a}, Y = y + \frac{b^2 - 4ac}{4a}$$

this is a parabola $Y = aX^2$

This is a parabola with vertex

$$\left(-\frac{b}{2a}, \frac{b^2 - 4ac}{4a} \right) \text{ or } \left(\frac{-b}{2a}, \frac{-D}{4a} \right) \text{ where}$$

$D = b^2 - 4ac$ and the parabola is opening upwards.

There are three possibilities.

For $a > 0$,

- i) If $D = b^2 - 4ac = 0$, the parabola touches x-axis and $y \geq 0$ for all x . e.g. $g(x) = x^2 - 2x + 1$
- ii) If $D = b^2 - 4ac > 0$, then parabola intersects x-axis at 2 distinct points. Here y is negative for values of x between the 2 roots and positive for large or small x .

- iii) If $D = b^2 - 4ac < 0$, the parabola lies above x-axis and $y \neq 0$ for any x . Here y is positive for all values of x . e.g. $f(x) = x^2 + 4x + 5$

iii) Cubic Function

Example : $f(x) = ax^3 + bx^2 + cx + d$ ($a \neq 0$)

Domain : \mathbb{R} or $(-\infty, \infty)$ and

Range : \mathbb{R} or $(-\infty, \infty)$

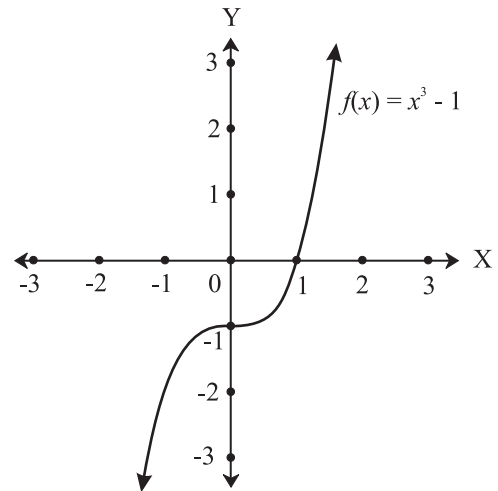


Fig. 6.24

Property:

- 1) Graph of $f(x) = x^3 - 1$

$f(x) = (x - 1)(x^2 + x + 1)$ cuts x-axis at only one point (1,0), which means $f(x)$ has one real root & two complex roots.

Note that, any polynomial of odd degree must have at least one real root, because the complex roots appear in conjugate pairs.

5) Radical Function

Ex: $f(x) = \sqrt[n]{x}$, $n \in \mathbb{N}$

1. Square root function

$$f(x) = \sqrt{x}, x \geq 0$$

(Since square root of negative number is not a real number, so the domain of \sqrt{x} is restricted to positive values of x).

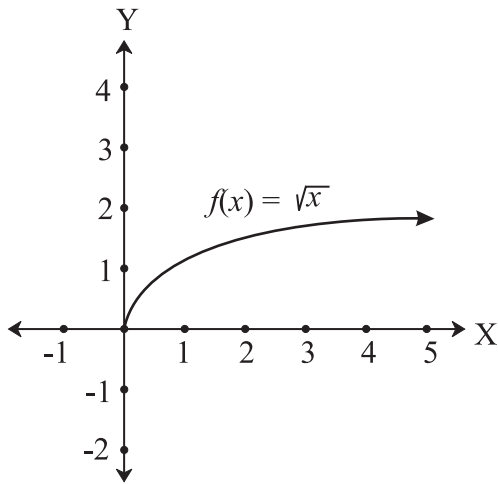


Fig. 6.25

Domain : $[0, \infty)$ and **Range :** $[0, \infty)$

Note :

- 1) If x is positive, there are two square roots of x . By convention \sqrt{x} is positive root and $-\sqrt{x}$ is the negative root.
- 2) If $-4 < x < 9$, as \sqrt{x} is only defined for $x \geq 0$, so $0 \leq \sqrt{x} < 3$.

Ex. 6 : Find the domain and range of $f(x) = \sqrt{9-x^2}$.

Soln. : $f(x) = \sqrt{9-x^2}$ is defined for $9-x^2 \geq 0$, i.e. $x^2-9 \leq 0$ i.e. $(x-3)(x+3) \leq 0$

Therefore $[-3, 3]$ is domain of $f(x)$. (Verify !)

To find range, let $\sqrt{9-x^2} = y$

Since square root is always positive, so $y \geq 0$... (I)

Also, on squaring we get $9-x^2 = y^2$

Since, $-3 \leq x \leq 3$

i.e. $0 \leq x^2 \leq 9$

i.e. $0 \geq -x^2 \geq -9$

i.e. $9 \geq 9-x^2 \geq 9-9$

i.e. $9 \geq 9-x^2 \geq 0$

i.e. $3 \geq \sqrt{9-x^2} \geq 0$

$\therefore 3 \geq y \geq 0$... (II)

From (I) and (II), $y \in [0,3]$ is range of $f(x)$.

2. Cube root function

$$f(x) = \sqrt[3]{x},$$

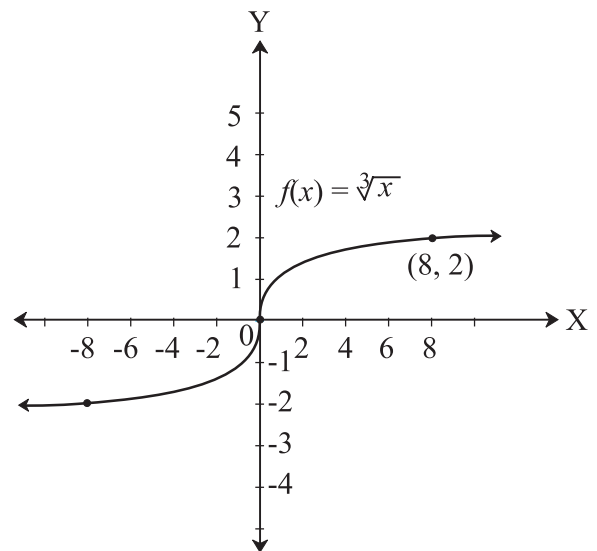


Fig. 6.26

Domain : \mathbb{R} and **Range :** \mathbb{R}

Note : If $-8 \leq x \leq 1$ then $-2 \leq \sqrt[3]{x} \leq 1$.

Ex. 7 : Find the domain $f(x) = \sqrt{x^3-8}$.

Soln. : $f(x)$ is defined for $x^3-8 \geq 0$

i.e. $x^3-2^3 \geq 0$, $(x-2)(x^2+2x+4) \geq 0$

In x^2+2x+4 , $a=1 > 0$ and $D=b^2-4ac = 2^2-4 \times 1 \times 4 = -12 < 0$

Therefore, x^2+2x+4 is a positive quadratic.

i.e. $x^2+2x+4 > 0$ for all x

Therefore $x-2 \geq 0$, $x \geq 2$ is the domain.

i.e. Domain is $x \in [2, \infty)$

6) Rational Function

Definition: Given polynomials

$p(x), q(x)$ $f(x) = \frac{p(x)}{q(x)}$ is defined for x if $q(x) \neq 0$.

Example : $f(x) = \frac{1}{x}, x \neq 0$

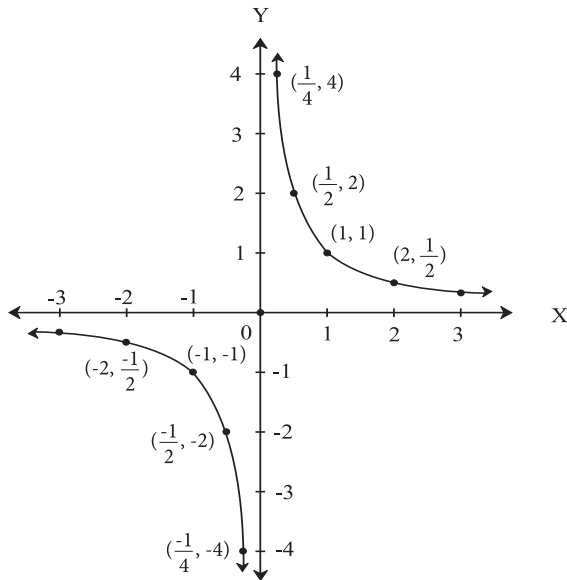


Fig. 6.27

Domain : $R - \{0\}$ and **Range :** $R - \{0\}$

Properties:

- 1) As $x \rightarrow 0$ i.e. (As x approaches 0) $f(x) \rightarrow \infty$ or $f(x) \rightarrow -\infty$, so the line $x = 0$ i.e. y -axis is called vertical asymptote. (A straight line which does not intersect the curve but as x approaches to ∞ or $-\infty$ the distance between the line and the curve tends to 0, is called an asymptote of the curve.)
- 2) As $x \rightarrow \infty$ or $x \rightarrow -\infty, f(x) \rightarrow 0, y = 0$ the line i.e. y -axis is called horizontal asymptote.
- 3) The domain of rational function $f(x) = \frac{p(x)}{q(x)}$ is all the real values of x except the zeroes of $q(x)$.

Ex. 8 : Find domain and range of the function

$$f(x) = \frac{6 - 4x^2}{4x + 5}$$

Solution : $f(x)$ is defined for all $x \in R$ except when denominator is 0.

$$\text{Since, } 4x + 5 = 0 \Rightarrow x = -\frac{5}{4}$$

So Domain of $f(x)$ is $R - \left\{-\frac{5}{4}\right\}$.

$$\text{To find the range, let } y = \frac{6 - 4x^2}{4x + 5}$$

$$\text{i.e. } y(4x + 5) = 6 - 4x^2$$

$$\text{i.e. } 4x^2 + (4y)x + 5y - 6 = 0.$$

This is a quadratic equation in x with y as constant.

Since $x \in R - \{-5/4\}$, i.e. x is real, we get

Solution if, $D = b^2 - 4ac \geq 0$

$$\text{i.e. } (4y)^2 - 4(4)(5y - 6) \geq 0$$

$$16y^2 - 16(5y - 6) \geq 0$$

$$y^2 - 5y + 6 \geq 0$$

$$(y - 2)(y - 3) \geq 0$$

Therefore $y \leq 2$ or $y \geq 3$ (Verify!)

Range of $f(x)$ is $(-\infty, 2] \cup [3, \infty)$

7) Exponential Function

Form : $f(x) = a^x$ is an exponential function with base a and exponent (or index) $x, a \neq 0$,

$a > 0$ and $x \in R$.

Example : $f(x) = 2^x$ and $f(x) = 2^{-x}$

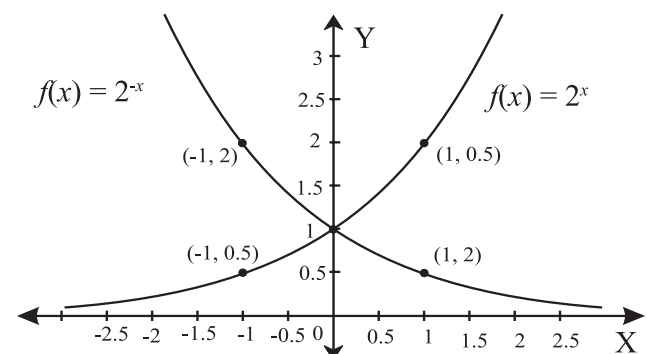


Fig. 6.28

Domain: R and **Range :** $(0, \infty)$

Properties:

- As $x \rightarrow -\infty$, then $f(x) = 2^x \rightarrow 0$, so the graph has horizontal asymptote ($y = 0$)
- By taking the natural base $e (\approx 2.718)$, graph of $f(x) = e^x$ is similar to that of 2^x in appearance

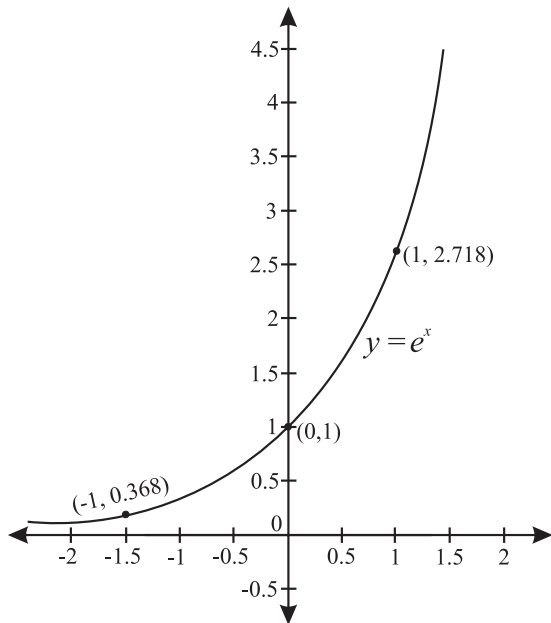


Fig. 6.29

- For $a > 0, a \neq 1$, if $a^x = a^y$ then $x = y$. So a^x is one-one function. (check graph for horizontal line test).
- $r > 1, m > n \Rightarrow r^m > r^n$ and
 $r < 1, m > n \Rightarrow r^m < r^n$

Ex. 9 : Solve $5^{2x+7} = 125$.

Solution : As $5^{2x+7} = 125$

$$\text{i.e. } 5^{2x+7} = 5^3, \therefore 2x + 7 = 3$$

$$\text{and } x = \frac{3-7}{2} = \frac{-4}{2} = -2$$

Ex. 10 : Find the domain of $f(x) = \sqrt{6 - 2^x - 2^{3-x}}$

Solution : Since \sqrt{x} is defined for $x \geq 0$

$$f(x) \text{ is defined for } 6 - 2^x - 2^{3-x} \geq 0$$

$$\text{i.e. } 6 - 2^x - \frac{2^3}{2^x} \geq 0$$

$$\text{i.e. } 6 \cdot 2^x - (2^x)^2 - 8 \geq 0$$

$$\text{i.e. } (2^x)^2 - 6 \cdot 2^x + 8 \leq 0$$

$$\text{i.e. } (2^x - 4)(2^x - 2) \leq 0$$

$$2^x \geq 2 \text{ and } 2^x \leq 4 \text{ (Verify !)}$$

$$2^x \geq 2^1 \text{ and } 2^x \leq 2^2$$

$$x \geq 1 \text{ and } x \leq 2 \text{ or } 1 \leq x \leq 2$$

Domain is $[1, 2]$

8) Logarithmic Function:

Let, $a > 0, a \neq 1$, Then logarithmic function $\log_a x$, $y = \log_a x$ if $x = a^y$.

for $x > 0$, is defined as

$$y = \log_a x \Leftrightarrow a^y = x$$

log arithmetic form exponential form

Properties:

- As $a^0 = 1$, so $\log_a 1 = 0$ and as $a^1 = a$,
so $\log_a a = 1$
- As $a^x = a^y \Leftrightarrow x = y$ so $\log_a x = \log_a y \Leftrightarrow x = y$
- Product rule of logarithms.
For $a, b, c > 0$ and $a \neq 1$,
 $\log_a bc = \log_a b + \log_a c$ (Verify !)
- Quotient rule of logarithms.
For $a, b, c > 0$ and $a \neq 1$,
 $\log_a \frac{b}{c} = \log_a b - \log_a c$ (Verify !)
- Power/Exponent rule of logarithms.
For $a, b, c > 0$ and $a \neq 1$,
 $\log_a b^c = c \log_a b$ (Verify !)

- 6) For natural base e , $\log_e x = \ln x$ as Natural Logarithm Function.

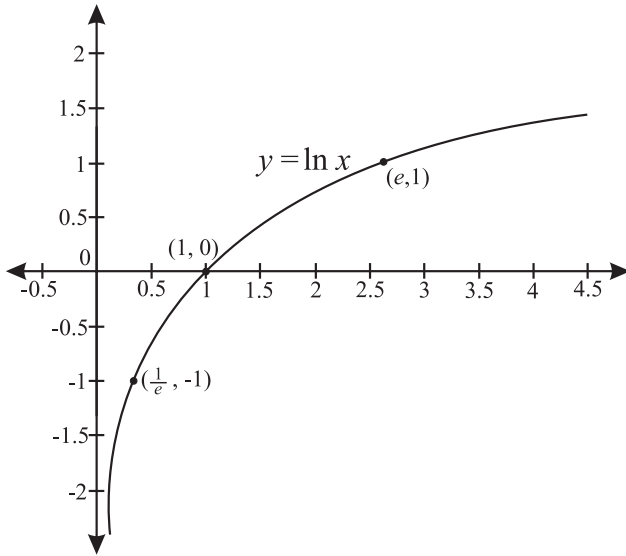


Fig. 6.30

Here domain of $\ln x$ is $(0, \infty)$ and range is $(-\infty, \infty)$.

- 8) Logarithmic inequalities:

- (i) If $a > 1$, $0 < m < n$ then $\log_a m < \log_a n$
e.g. $\log_{10} 20 < \log_{10} 30$
- (ii) If $0 < a < 1$, $0 < m < n$ then $\log_a m > \log_a n$
e.g. $\log_{0.1} 20 > \log_{0.1} 30$
- (iii) For $a, m > 0$ if a and m lies on the same side of unity (i.e. 1) then $\log_a m > 0$.
e.g. $\log_2 3 > 0$, $\log_{0.3} 0.5 > 0$
- (iv) For $a, m > 0$ if a and m lies on the different sides of unity (i.e. 1) then $\log_a m > 0$.
e.g. $\log_{0.2} 3 < 0$, $\log_3 0.5 > 0$

Ex. 11 : Write $\log 72$ in terms of $\log 2$ and $\log 3$.

Solution : $\log 72 = \log(2^3 \cdot 3^2)$
 $= \log 2^3 + \log 3^2$ (\because Power rule)
 $= 3 \log 2 + 2 \log 3$ (\because Power rule)

Ex. 12 : Evaluate $\ln e^9 - \ln e^4$.

Solution : $\ln e^9 - \ln e^4 = \log_e e^9 - \log_e e^4$
 $= 9 \log_e e - 4 \log_e e$
 $= 9(1) - 4(1)$ ($\because \ln e = 1$)
 $= 5$

Ex. 13 : Expand $\log \left[\frac{x^3(x+3)}{2(x-4)^2} \right]$

Solution : Using Quotient rule
 $= \log [x^3(x+3)] - \log [2(x-4)^2]$
 Using Product rule
 $= [\log x^3 + \log(x+3)] - [\log 2 + \log(x-4)^2]$
 Using Power rule
 $= [3 \log x + \log(x+3)] - [\log 2 + 2 \log(x-4)]$
 $= 3 \log x + \log(x+3) - \log 2 + 2 \log(x-4)$

Ex. 14 : Combine

$3 \ln(p+1) - \frac{1}{2} \ln r + 5 \ln(2q+3)$ into single logarithm.

Solution : Using Power rule,
 $= \ln(p+1)^3 - \ln r^{\frac{1}{2}} + \ln(2q+3)^5$
 Using Quotient rule
 $= \ln \frac{(p+1)^3}{\sqrt{r}} + \ln(2q+3)^5$
 Using Product rule
 $= \ln \left[\frac{(p+1)^3}{\sqrt{r}} (2q+3)^5 \right]$

Ex. 15 : Find the domain of $\ln(x-5)$.

Solution : As $\ln(x-5)$ is defined for $(x-5) > 0$ that is $x > 5$ so domain is $(5, \infty)$.

Let's note:

- 1) $\log(x + y) \neq \log x + \log y$
- 2) $\log x \log y \neq \log(xy)$
- 3) $\frac{\log x}{\log y} \neq \log\left(\frac{x}{y}\right)$
- 4) $(\log x)^n \neq n \log^n$

9) Change of base formula:

For $a, x, b > 0$ and $a, b \neq 1$, $\log_a x = \frac{\log_b x}{\log_b a}$

Note: $\log_a x = \frac{\log_x x}{\log_x a} = \frac{1}{\log_x a}$ (Verify !)

Ex. 16 : Evaluate $\frac{\log_4 81}{\log_4 9}$

Solution : By Change of base law, as the base is same (that is 4)

$$\frac{\log_4 81}{\log_4 9} = \log_9 81 = 2$$

Ex. 17 : Prove that, $2\log_b a^4 \cdot \log_c b^3 \cdot \log_a c^5 = 120$

Solution : L.H.S. = $2\log_b a^4 \cdot \log_c b^3 \cdot \log_a c^5$
 $= 4 \times 2\log_b a \times 3\log_c b \times 5\log_a c$

Using change of base law,

$$= 4 \times 2 \frac{\log a}{\log b} \times 3 \frac{\log b}{\log c} \times 5 \frac{\log c}{\log a}$$

$$= 120$$

Ex. 18 : Find the domain of $f(x) = \log_{x+5}(x^2 - 4)$

Solution : Since $\log_a x$ is defined for $a, x > 0$ and $a \neq 1$ $f(x)$ is defined for $(x^2 - 4) > 0$, $x + 5 > 0$, $x + 5 \neq 1$.

i.e. $(x - 2)(x + 2) > 0$, $x > -5$, $x \neq -4$

i.e. $x < -2$ or $x > 2$ and $x > -5$ and $x \neq -4$

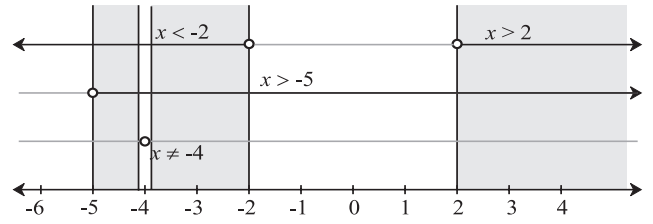


Fig. 6.31

9) Trigonometric function

The graphs of trigonometric functions are discussed in chapter 2 of Mathematics Book I.

$f(x)$	Domain	Range
$\sin x$	\mathbb{R}	$[-1, 1]$
$\cos x$	\mathbb{R}	$[-1, 1]$
$\tan x$	$\mathbb{R} - \left\{ \pm \frac{\pi}{2}, \pm \frac{3\pi}{2}, \dots \right\}$	\mathbb{R}

EXERCISE 6.1

1) Check if the following relations are functions.

(a)

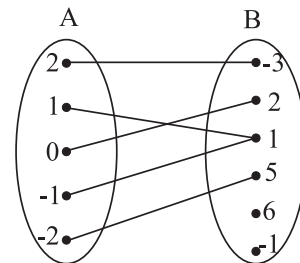


Fig. 6.32

(b)

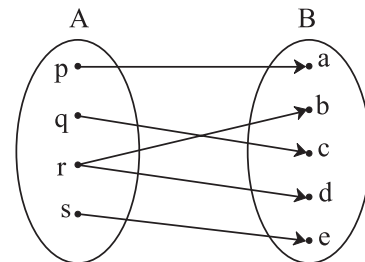


Fig. 6.33

(c)

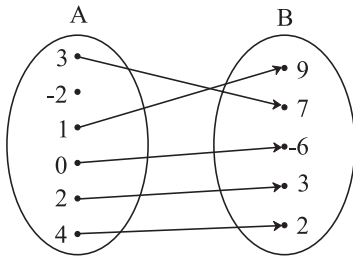


Fig. 6.34

2) Which sets of ordered pairs represent functions from $A = \{1, 2, 3, 4\}$ to $B = \{-1, 0, 1, 2, 3\}$? Justify.

- (a) $\{(1,0), (3,3), (2,-1), (4,1), (2,2)\}$
- (b) $\{(1,2), (2,-1), (3,1), (4,3)\}$
- (c) $\{(1,3), (4,1), (2,2)\}$
- (d) $\{(1,1), (2,1), (3,1), (4,1)\}$

3) Check if the relation given by the equation represents y as function of x .

- (a) $2x + 3y = 12$
- (b) $x + y^2 = 9$
- (c) $x^2 - y = 25$
- (d) $2y + 10 = 0$
- (e) $3x - 6 = 21$

4) If $f(m) = m^2 - 3m + 1$, find

- (a) $f(0)$
- (b) $f(-3)$
- (c) $f\left(\frac{1}{2}\right)$
- (d) $f(x+1)$
- (e) $f(-x)$
- (f) $\left(\frac{f(2+h) - f(2)}{h}\right), h \neq 0$.

5) Find x , if $g(x) = 0$ where

- (a) $g(x) = \frac{5x-6}{7}$
- (b) $g(x) = \frac{18-2x^2}{7}$
- (c) $g(x) = 6x^2 + x - 2$
- (d) $g(x) = x^3 - 2x^2 - 5x + 6$

6) Find x , if $f(x) = g(x)$ where

- (a) $f(x) = x^4 + 2x^2, g(x) = 11x^2$
- (b) $f(x) = \sqrt{x} - 3, g(x) = 5 - x$

7) If $f(x) = \frac{a-x}{b-x}$, $f(2)$ is undefined, and $f(3) = 5$, find a and b .

8) Find the domain and range of the following functions.

(a) $f(x) = 7x^2 + 4x - 1$

(b) $g(x) = \frac{x+4}{x-2}$

(c) $h(x) = \frac{\sqrt{x+5}}{5+x}$

(d) $f(x) = \sqrt[3]{x+1}$

(e) $f(x) = \sqrt{(x-2)(5-x)}$

(f) $f(x) = \sqrt{\frac{x-3}{7-x}}$

(g) $f(x) = \sqrt{16-x^2}$

9) Express the area A of a square as a function of its (a) side s (b) perimeter P .

10) Express the area A of circle as a function of its (a) radius r (b) diameter d (c) circumference C .

11) An open box is made from a square of cardboard of 30 cms side, by cutting squares of length x centimeters from each corner and folding the sides up. Express the volume of the box as a function of x . Also find its domain.

Let f be a subset of $Z \times Z$ defined by

12) $f = \{(ab, a+b) : a, b \in Z\}$. Is f a function from Z to Z ? Justify.

14) Check the injectivity and surjectivity of the following functions.

(a) $f: N \rightarrow N$ given by $f(x) = x^2$

(b) $f: Z \rightarrow Z$ given by $f(x) = x^2$

(c) $f: R \rightarrow R$ given by $f(x) = x^2$

- (d) $f: \mathbb{N} \rightarrow \mathbb{N}$ given by $f(x) = x^3$
 (e) $f: \mathbb{R} \rightarrow \mathbb{R}$ given by $f(x) = x^3$
- 14) Show that if $f: A \rightarrow B$ and $g: B \rightarrow C$ are one-one, then $g \circ f$ is also one-one.
- 15) Show that if $f: A \rightarrow B$ and $g: B \rightarrow C$ are onto, then $g \circ f$ is also onto.
- 16) If $f(x) = 3(4^{x+1})$ find $f(-3)$.
- 17) Express the following exponential equations in logarithmic form
- (a) $2^5 = 32$ (b) $54^0 = 1$
 (c) $23^1 = 23$ (d) $9^{3/2} = 27$
 (e) $3^{-4} = \frac{1}{81}$ (f) $10^{-2} = 0.01$
 (g) $e^2 = 7.3890$ (h) $e^{1/2} = 1.6487$
 (i) $e^{-x} = 6$
- 18) Express the following logarithmic equations in exponential form
- (a) $\log_2 64 = 6$ (b) $\log_5 \frac{1}{25} = -2$
 (c) $\log_{10} 0.001 = -3$ (d) $\log_{1/2} (-8) = 3$
 (e) $\ln 1 = 0$ (f) $\ln e = 1$
 (g) $\ln \frac{1}{2} = -0.693$
- 19) Find the domain of
- (a) $f(x) = \ln(x-5)$
 (b) $f(x) = \log_{10}(x^2 - 5x + 6)$
- 20) Write the following expressions as sum or difference of logarithms
- (a) $\log \left(\frac{pq}{rs} \right)$ (b) $\log (\sqrt{x} \sqrt[3]{y})$
 (c) $\ln \left(\frac{a^3(a-2)^2}{\sqrt{b^2+5}} \right)$
 (d) $\ln \left[\frac{\sqrt[3]{x-2}(2x+1)^4}{(x+4)\sqrt{2x+4}} \right]^2$
- 21) Write the following expressions as a single logarithm.
- (a) $5\log x + 7\log y - \log z$
 (b) $\frac{1}{3} \log(x-1) + \frac{1}{2} \log(x)$
 (c) $\ln(x+2) + \ln(x-2) - 3\ln(x+5)$
- 22) Given that $\log 2 = a$ and $\log 3 = b$, write $\log \sqrt{96}$ in terms of a and b .
- 23) Prove that
- (a) $b^{\log_b a} = a$ (b) $\log_{b^m} a = \frac{1}{m} \log_b a$
 (c) $a^{\log_c b} = b^{\log_c a}$
- 24) If $f(x) = ax^2 - bx + 6$ and $f(2) = 3$ and $f(4) = 30$, find a and b
- 25) Solve for x .
- (a) $\log 2 + \log(x+3) - \log(3x-5) = \log 3$
 (b) $2\log_{10} x = 1 + \log_{10} \left(x + \frac{11}{10} \right)$
 (c) $\log_2 x + \log_4 x + \log_{16} x = \frac{21}{4}$
 (d) $x + \log_{10}(1+2^x) = x \log_{10} 5 + \log_{10} 6$
- 26) If $\log \left(\frac{x+y}{3} \right) = \frac{1}{2} \log x + \frac{1}{2} \log y$, show that $\frac{x}{y} + \frac{y}{x} = 7$.
- 27) If $\log \left(\frac{x-y}{4} \right) = \log \sqrt{x} + \log \sqrt{y}$, show that $(x+y)^2 = 20xy$
- 28) If $x = \log_a bc$, $y = \log_b ca$, $z = \log_c ab$ then prove that $\frac{1}{1+x} + \frac{1}{1+y} + \frac{1}{1+z} = 1$

6.2 Algebra of functions:

Let f and g be functions with domains A and B . Then the functions $f + g, f - g, fg, \frac{f}{g}$ are defined on $A \cap B$ as follows.

Operations
$(f + g)(x) = f(x) + g(x)$
$(f - g)(x) = f(x) - g(x)$
$(f \cdot g)(x) = f(x) \cdot g(x)$
$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$ where $g(x) \neq 0$

Ex. 1 : If $f(x) = x^2 + 2$ and $g(x) = 5x - 8$, then find

- $(f + g)(1)$
- $(f - g)(-2)$
- $(f \circ g)(3m)$
- $\frac{f}{g}(0)$

Solution : i) As $(f + g)(x) = f(x) + g(x)$

$$\begin{aligned}(f + g)(1) &= f(1) + g(1) \\ &= [(1)^2 + 2] + [5(1) - 8] \\ &= 3 + (-3) \\ &= 0\end{aligned}$$

ii) As $(f - g)(x) = f(x) - g(x)$

$$\begin{aligned}(f - g)(-2) &= f(-2) - g(-2) \\ &= [(-2)^2 + 2] - [5(-2) - 8] \\ &= [4 + 2] - [-10 - 8] \\ &= 6 + 18 \\ &= 24\end{aligned}$$

iii) As $(fg)(x) = f(x)g(x)$

$$\begin{aligned}(f \circ g)(3m) &= f(3m)g(3m) \\ &= [(3m)^2 + 2][5(3m) - 8] \\ &= [9m^2 + 2][15m - 8] \\ &= 135m^3 - 72m^2 + 30m - 16\end{aligned}$$

iv) As $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}, g(x) \neq 0$

$$\begin{aligned}\left(\frac{f}{g}\right)(0) &= \frac{f(0)}{g(0)} = \frac{0^2 + 2}{5(0) - 8} \\ &= \frac{2}{-8} = -\frac{1}{4}\end{aligned}$$

Ex. 2 : Given the function $f(x) = 5x^2$ and

$g(x) = \sqrt{4-x}$ find the domain of

- $(f + g)(x)$
- $(f \circ g)(x)$
- $\frac{f}{g}(x)$

Solution : i) Domain of $f(x) = 5x^2$ is $(-\infty, \infty)$.

To find domain of $g(x) = \sqrt{4-x}$

$$\begin{aligned}4 - x &\geq 0 \\ x - 4 &\leq 0\end{aligned}$$

Let $x \leq 4$, So domain is $(-\infty, 4]$.

Therefore, domain of $(f + g)(x)$ is $(-\infty, \infty) \cap (-\infty, 4]$, that is $(-\infty, 4]$

ii) Similarly, domain of $(f \circ g)(x) = 5x^2\sqrt{4-x}$ is $(-\infty, 4]$

iii) And domain of $\left(\frac{f}{g}\right)(x) = \frac{5x^2}{\sqrt{4-x}}$ is $(-\infty, 4)$

As, at $x = 4$ the denominator $g(x) = 0$.

6.2.1 Composition of Functions:

A method of combining the function $f: A \rightarrow B$ with $g: B \rightarrow C$ is composition of functions, defined as $(f \circ g)(x) = f[g(x)]$ an read as 'f composed with g'

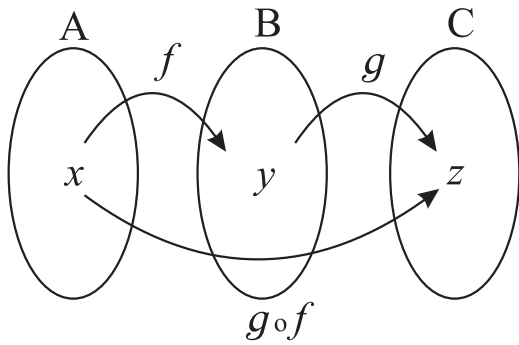


Fig. 6.35

Note:

- 1) The domain of $g \circ f$ is the set of all x in A such that $f(x)$ is in the B . The range of $g \circ f$ is set of all $g[f(x)]$ in C such that $f(x)$ is in B .
- 2) Domain of $g \circ f \subseteq$ Domain of f and Range of $g \circ f \subseteq$ Range of g .

Illustration:

A cow produces 4 liters of milk in a day. Then x number of cows produce $4x$ liters of milk in a day. This is given by function $f(x) = 4x = 'y'$. Price of one liter milk is Rs. 50. Then the price of y liters of the milk is Rs. $50y$. This is given by another function $g(y) = 50y$. Now a function $h(x)$ gives the money earned from x number of cows in a day as a composite function of f and g as $h(x) = (g \circ f)(x) = g[f(x)] = g(4x) = 50(4x) = 200x$.

Ex. 3 : If $f(x) = \frac{2}{x+5}$ and $g(x) = x^2 - 1$, then find

- i) $(f \circ g)(x)$ ii) $(g \circ f)(3)$

Solution :

i) As $(f \circ g)(x) = f[g(x)]$ and $f(x) = \frac{2}{x+5}$

Replace x from $f(x)$ by $g(x)$, to get

$$\begin{aligned} (f \circ g)(x) &= \frac{2}{g(x)+5} \\ &= \frac{2}{x^2-1+5} \\ &= \frac{2}{x^2+4} \end{aligned}$$

ii) As $(g \circ f)(x) = g[f(x)]$ and $g(x) = x^2 - 1$
Replace x by $f(x)$, to get

$$\begin{aligned} (g \circ f)(x) &= [f(x)]^2 - 1 \\ &= \left(\frac{2}{x+5}\right)^2 - 1 \end{aligned}$$

Now let $x = 3$

$$\begin{aligned} (g \circ f)(3) &= \left(\frac{2}{3+5}\right)^2 - 1 \\ &= \left(\frac{2}{8}\right)^2 - 1 \\ &= \left(\frac{1}{4}\right)^2 - 1 \\ &= \frac{1-16}{16} \\ &= -\frac{15}{16} \end{aligned}$$

Ex 4 : If $f(x) = x^2$, $g(x) = x + 5$, and $h(x) = \frac{1}{x}$, $x \neq 0$, find $(g \circ f \circ h)(x)$

Solution : $(g \circ f \circ h)(x)$

$$\begin{aligned} &= g\{f[h(x)]\} \\ &= g\left[f\left(\frac{1}{x}\right)\right] \\ &= g\left[\left(\frac{1}{x}\right)^2\right] \\ &= \left(\frac{1}{x}\right)^2 + 5 \\ &= \frac{1}{x^2} + 5 \end{aligned}$$

Ex. 5 : If $h(x) = (x - 5)^2$, find the functions f and g , such that $h = f \circ g$.

→ In $h(x)$, 5 is subtracted from x first and then squared. Let $g(x) = x - 5$ and $f(x) = x^2$, (verify)

Ex. 6 : Express $m(x) = \frac{1}{x^3+7}$ in the form of $f \circ g \circ h$

→ In $m(x)$, x is cubed first then 7 is added and then its reciprocal taken. So,

$$h(x) = x^3, g(x) = x + 7 \text{ and } f(x) = \frac{1}{x}, \text{ (verify)}$$

6.2.2 Inverse functions:

Let $f: A \rightarrow B$ be one-one and onto function and $f(x) = y$ for $x \in A$. The inverse function

$f^{-1}: B \rightarrow A$ is defined as $f^{-1}(y) = x$ if $f(x) = y$

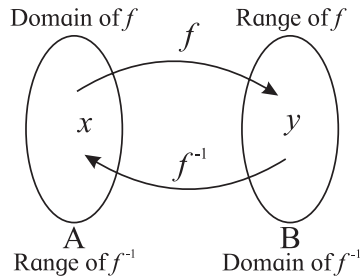


Fig. 6.36

Note:

- As f is one-one and onto every element $y \in B$ has a unique element $x \in A$ such that $y = f(x)$.
- If f and g are one-one and onto functions such that $f[g(x)] = x$ for every $x \in \text{Domain of } g$ and $g[f(x)] = x$ for every $x \in \text{Domain of } f$, then g is called inverse of function f . Function g is denoted by f^{-1} (read as f inverse).
i.e. $f[g(x)] = g[f(x)] = x$ then $g = f^{-1}$ which
Moreover this means $f[f^{-1}(x)] = f^{-1}[f(x)] = x$
- $f^{-1}(x) \neq [f(x)]^{-1}$, because $[f(x)]^{-1} = \frac{1}{f(x)}$
 $[f(x)]^{-1}$ is reciprocal of function $f(x)$ where as
 $f^{-1}(x)$ is the inverse function of $f(x)$.

e.g. If f is one-one onto function with $f(3) = 7$ then $f^{-1}(7) = 3$.

Ex. 7 : If f is one-one onto function with $f(x) = 9 - 5x$, find $f^{-1}(-1)$.

Soln. : \rightarrow Let $f^{-1}(-1) = m$, then $-1 = f(m)$

Therefore,

$$-1 = 9 - 5m$$

$$5m = 9 + 1$$

$$5m = 10$$

$$m = 2$$

That is $f(2) = -1$, so $f^{-1}(-1) = 2$.

Ex. 8 : Verify that $f(x) = \frac{x-5}{8}$ and $g(x) = 8x + 5$

are inverse functions of each other.

Solution : As $f(x) = \frac{x-5}{8}$, replace x in $f(x)$ with $g(x)$

$$f[g(x)] = \frac{g(x)-5}{8} = \frac{8x+5-5}{8} = \frac{8x}{8} = x$$

and $g(x) = 8x + 5$, replace x in $g(x)$ with $f(x)$

$$g[f(x)] = 8f(x) + 5 = 8 \left[\frac{x-5}{8} \right] + 5 = x - 5 + 5 = x$$

As $f[g(x)] = x$ and $g[f(x)] = x$, f and g are inverse functions of each other.

Ex. 9 : Determine whether the function

$$f(x) = \frac{2x+1}{x-3} \text{ has inverse, if it exists find it.}$$

Solution : f^{-1} exists only if f is one-one and onto.

$$\text{Consider } f(x_1) = f(x_2),$$

Therefore,

$$\frac{2x_1+1}{x_1-3} = \frac{2x_2+1}{x_2-3}$$

$$(2x_1+1)(x_2-3) = (2x_2+1)(x_1-3)$$

$$2x_1x_2 - 6x_1 + x_2 - 3 = 2x_1x_2 - 6x_2 + x_1 - 3$$

$$-6x_1 + x_2 = -6x_2 + x_1$$

$$6x_1 + x_2 = 6x_2 + x_1$$

$$7x_2 = 7x_1$$

$$x_2 = x_1$$

Hence, f is one-one function.

$$\text{Let } f(x) = y, \text{ so } y = \frac{2x+1}{x-3}$$

Express x as function of y , as follows

$$y = \frac{2x+1}{x-3}$$

$$y(x-3) = 2x+1$$

$$xy - 3y = 2x + 1$$

$$xy - 2x = 3y + 1$$

$$x(y - 2) = 3y + 1$$

$$\therefore x = \frac{3y+1}{y-2} \text{ for } y \neq 2.$$

Thus for any $y \neq 2$,

we have x such that $f(x) = y$

f^{-1} is well defined on $\mathbb{R} - \{2\}$

If $f(x) = 2$ i.e. $2x + 1 = 2(x - 3)$

i.e. $2x + 1 = 2x - 6$ i.e. $1 = -6$

Which is contradiction.

$2 \notin \text{Range of } f$.

Here the range of $f(x)$ is $\mathbb{R} - \{2\}$.

x is defined for all y in the range.

Therefore $f(x)$ is onto function.

As f is one-one and onto, so f^{-1} exists.

As $f(x) = y$, so $f^{-1}(y) = x$

Therefore, $f^{-1}(y) = \frac{3y+1}{y-2}$

Replace x by y , to get

$$f^{-1}(x) = \frac{3x+1}{x-2}.$$

6.2.3 Piecewise Defined Functions:

A function defined by two or more equations on different parts of the given domain is called piecewise defined function.

$$\text{e.g.: If } f(x) = \begin{cases} x+1 & \text{if } x < 1 \\ 4-x & \text{if } x \geq 1 \end{cases}$$

Here $f(3) = 4 - 3 = 1$ as $3 > 1$,

whereas $f(-2) = -2 + 1 = -1$ as $-2 < 1$ and

$f(1) = 4 - 1 = 3$.

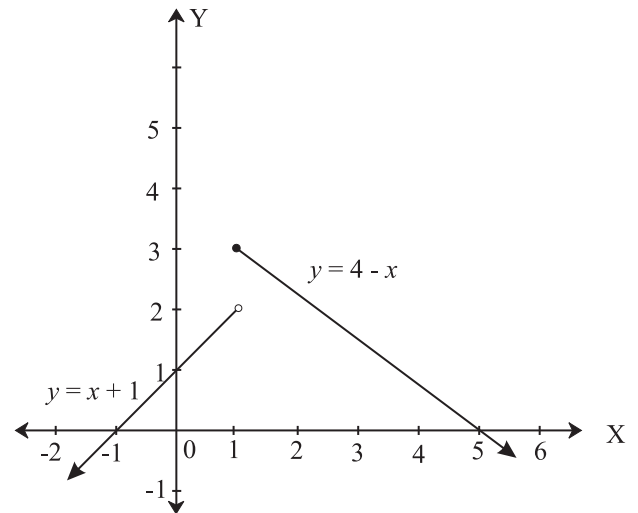


Fig. 6.37

As $(1,3)$ lies on line $y = 4 - x$, so it is shown by small black disc on that line. $(1,2)$ is shown by small white disc on the line $y = x + 1$, because it is not on the line.

1) Signum function :

Definition: $f(x) = \text{sgn}(x)$ is a piecewise function

$$f(x) = \begin{cases} -1 & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ 1 & \text{if } x > 0 \end{cases}$$

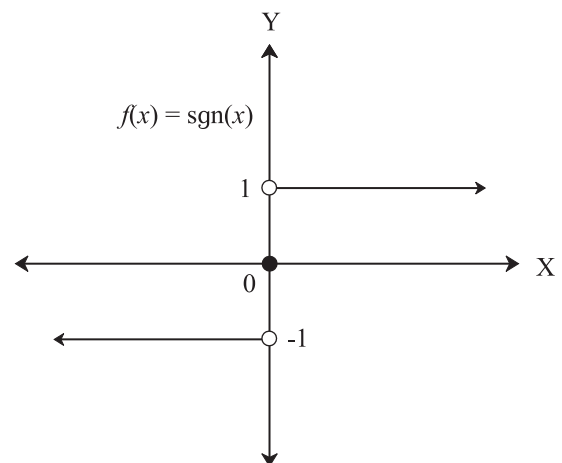


Fig. 6.38

Domain: \mathbb{R} and **Range:** $\{-1, 0, 1\}$

Properties:

- 1) For $x > 0$, the graph is line $y = 1$ and for $x < 0$, the graph is line $y = -1$.
- 2) For $f(0) = 0$, so point $(0,0)$ is shown by black disc, whereas points $(0,-1)$ and $(0,1)$ are shown by white discs.

Absolute value function (Modulus function):

Definition: $f(x) = |x|$, is a piece wise function

$$f(x) = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

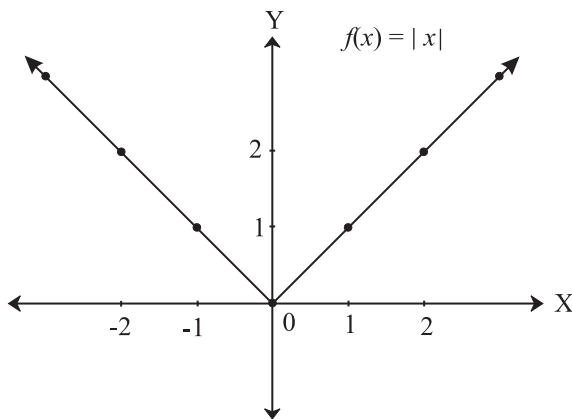


Fig. 6.39

Domain : \mathbb{R} or $(-\infty, \infty)$ and **Range :** $[0, \infty)$

Properties:

- 1) Graph of $f(x) = |x|$ is union of line $y = x$ from quadrant I with the line $y = -x$ from quadrant II. As origin marks the change of directions of the two lines, we call it a critical point.
- 2) Graph is symmetric about y-axis.
- 3) Graph of $f(x) = |x-3|$ is the graph of $|x|$ shifted 3 units right and the critical point is $(3,0)$.
- 4) $f(x) = |x|$, represents the distance of x from origin.
- 5) If $|x| = m$, then it represents every x whose distance from origin is m , that is $x = +m$ or $x = -m$.

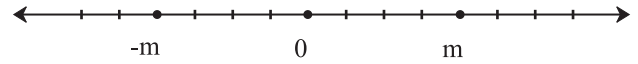


Fig. 6.40

- 6) If $|x| < m$, then it represents every x whose distance from origin is less than m , $0 \leq x < m$ and $0 \geq x > -m$ That is $-m < x < m$. In interval notation $x \in (-m, m)$



Fig. 6.41

- 7) If $|x| \geq m$, then it represents every x whose distance from origin is greater than or equal to m , so, $x \geq m$ and $x \leq -m$. In interval notation $x \in (-\infty, m] \cup [m, \infty)$

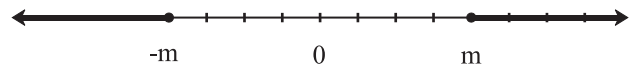


Fig. 6.42

- 8) If $m < |x| < n$, then it represents all x whose distance from origin is greater than m but less than n . That is $x \in (-n, -m) \cup (m, n)$.

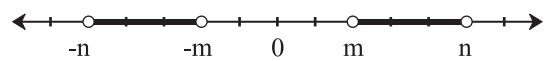


Fig. 6.43

- 9) Triangle inequality $|x + y| \leq |x| + |y|$.
Verify by taking different values for x and y (positive or negative).
- 10) $|x|$ can also be defined as $|x| = \sqrt{x^2} = \max\{x, -x\}$.

Ex. 10 : Solve $|4x - 5| \leq 3$.

Solution : If $|x| \leq m$, then $-m \leq x \leq m$

Therefore

$$-3 \leq 4x - 5 \leq 3$$

$$-3 + 5 \leq 4x \leq 3 + 5$$

$$2 \leq 4x \leq 8$$

$$\frac{2}{4} \leq x \leq \frac{8}{4}$$

$$\frac{1}{2} \leq x \leq 2$$

Ex. 11 : Find the domain of $\frac{1}{\sqrt{||x|-1|-3}}$

Solution : As function is defined for $||x|-1|-3 > 0$

Therefore $||x|-1| > 3$

So $|x|-1 > 3$ or $|x|-1 < -3$

That is

$|x| > 3 + 1$ or $|x| < -3 + 1$

$|x| > 4$ or $|x| < -2$

But $|x| < -2$ is not possible as $|x| > 0$ always

So $-4 < x < 4$, $x \in (-4, 4)$.

Ex. 12 : Solve $|x - 1| + |x + 2| = 8$.

Solution : Let $f(x) = |x - 1| + |x + 2|$

Here the critical points are at $x = 1$ and $x = -2$.

They divide number line into 3 parts, as follows.

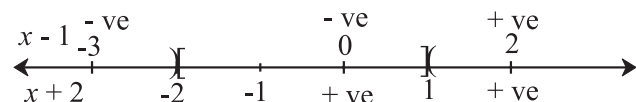


Fig. 6.44

Region	Test Value	Sign	$f(x)$
I $x < -2$	-3	$(x-1) < 0$, $(x+2) < 0$	$-(x-1) - (x+2)$ $= -2x - 1$
II $-2 \leq x \leq 1$	0	$(x-1) < 0$, $(x+2) > 0$	$-(x-1) + (x+2)$ $= 3$
III $x > 1$	2	$(x-1) > 0$, $(x+2) > 0$	$(x-1) + (x+2)$ $= 2x + 1$

As $f(x) = 8$

From I, $-2x - 1 = 8 \therefore -2x = 9 \therefore x = -\frac{9}{2}$.

From II, $3 = 8$, which is impossible, hence there is no solution in this region.

From III, $2x + 1 = 8 \therefore 2x = 7 \therefore x = \frac{7}{2}$.

Solutions are $x = -\frac{9}{2}$ and $x = \frac{7}{2}$.

3) Greatest Integer Function (Step Function):

Definition: For every real x , $f(x) = [x] =$ The greatest integer less than or equal to x . $[x]$ is also called as floor function and represented by $\lfloor x \rfloor$.

Illustrations:

1) $f(5.7) = [5.7] =$ greatest integer less than or equal to 5.7

Integers less than or equal to 5.7 are 5, 4, 3, 2 of which 5 is the greatest.

2) $f(-6.3) = [-6.3] =$ greatest integer less than or equal to -6.3.

Integers less than or equal to -6.3 are -10, -9, -8, -7 of which -7 is the greatest.

$\therefore [-6.3] = -7$

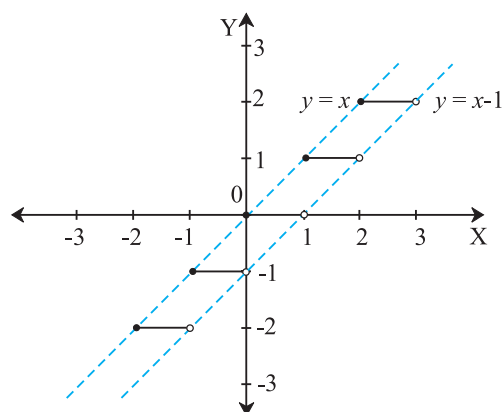
3) $f(2) = [2] =$ greatest integer less than or equal to 2 = 2.

4) $[\pi] = 3$ 5) $[e] = 2$

The function can be defined piece-wise as follows

$f(x) = n$, if $n \leq x < n + 1$ or $x \in [n, n + 1)$, $n \in \mathbb{I}$

$$f(x) = \begin{cases} -2 & \text{if } -2 \leq x < -1 \text{ or } x \in [-2, -1) \\ -1 & \text{if } -1 \leq x < 0 \text{ or } x \in [-1, 0) \\ 0 & \text{if } 0 \leq x < 1 \text{ or } x \in [0, 1) \\ 1 & \text{if } 1 \leq x < 2 \text{ or } x \in [1, 2) \\ 2 & \text{if } 2 \leq x < 3 \text{ or } x \in [2, 3) \end{cases}$$



Graph of $f(x) = [x]$

Fig. 6.45

Domain = \mathbb{R} and Range = \mathbb{I} (Set of integers)

Properties:

- 1) If $x \in [2,3), f(x) = 2$ shown by horizontal line. At exactly $x = 2, f(2) = 2, 2 \in [2,3)$ hence shown by black disc, whereas $3 \notin [2,3)$ hence shown by white disc.
- 2) Graph of $y = [x]$ lies in the region bounded by lines $y = x$ and $y = x - 1$. So $x - 1 \leq [x] < x$
- 3) $[x] + [-x] = \begin{cases} 0 & \text{if } x \in I \\ -1 & \text{if } x \notin I \end{cases}$

Ex. $[3.4] + [-3.4] = 3 + (-4) = -1$ where $3.4 \notin I$
 $[5] + [-5] = 5 + (-5) = 0$ where $5 \in I$

4) $[x+n] = [x] + n$, where $n \in I$

Ex. $[4.5 + 7] = [11.5] = 11$ and

$$[4.5] + 7 = 4 + 7 = 11$$

4) Fractional part function:

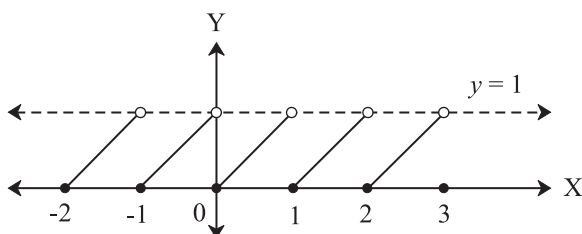
Definition: For every real $x, f(x) = \{x\}$ is defined as $\{x\} = x - [x]$

Illustrations:

$$f(4.8) = \{4.8\} = 4.8 - [4.8] = 4.8 - 4 = 0.8$$

$$f(-7.1) = \{-7.1\} = -7.1 - [-7.1] \\ = -7.1 - (-8) = -7.1 + 8 = 0.9$$

$$f(8) = \{8\} = 8 - [8] = 8 - 8 = 0$$



Graph of $f(x) = \{x\}$

Fig. 6.46

Domain = \mathbb{R} and Range = $[0,1)$

Properties:

- 1) If $x \in [0,1), f(x) = \{x\} \in [0,1)$ shown by slant line $y = x$. At $x = 0, f(0) = 0, 0 \in [0,1)$ hence shown by black disc, whereas at $x = 1, f(1) = 1, 1 \notin [0,1)$ hence shown by white disc.
- 2) Graph of $y = \{x\}$ lies in the region bounded by $y = 0$ and $y = 1$. So $0 \leq \{x\} < 1$
- 3) $\{x\} + \{-x\} = \begin{cases} 0 & \text{if } x \in I \\ 1 & \text{if } x \notin I \end{cases}$

Ex. 13: $\{5.2\} + \{-5.2\} = 0.2 + 0.8 = 1$ where $5.2 \in I$
 $\{7\} + \{-7\} = 0 + (0) = 0$ where $7 \in I$

4) $\{x \pm n\} = \{x\}$, where $n \in I$

Ex. 14 : $\{2.8+5\} = \{7.8\} = 0.8$ and $\{2.8\} = 0.8$
 $\{2.8 - 5\} = \{-2.2\} = -2.2 - (-2.2) = -2.2 - (-3) = 0.8$ ($\because \{x\} = x - [x]$)

Ex. 15 : If $\{x\}$ and $[x]$ are the fractional part function and greatest integer function of x respectively. Solve for x , if $\{x + 1\} + 2x = 4[x + 1] - 6$.

Solution : $\{x + 1\} + 2x = 4[x + 1] - 6$

Since $\{x + n\} = \{x\}$ and $[x + n] = [x] + n$, for $n \in I$, also $x = [x] + \{x\}$

$$\therefore \{x\} + 2(\{x\} + [x]) = 4([x] + 1) - 6$$

$$\therefore \{x\} + 2\{x\} + 2[x] = 4[x] + 4 - 6$$

$$\therefore 3\{x\} = 4[x] - 2[x] - 2$$

$$\therefore 3\{x\} = 2[x] - 2 \quad \dots (I)$$

Since $0 \leq \{x\} < 1$

$$\therefore 0 \leq 3\{x\} < 3$$

$$\therefore 0 \leq 2[x] - 2 < 3 \quad (\because \text{from I})$$

$$\therefore 0 + 2 \leq 2[x] < 3 + 2$$

$$\therefore 2 \leq 2[x] < 5$$

$$\therefore \frac{2}{2} \leq [x] < \frac{5}{2}$$

$$\therefore 1 \leq [x] < 2.5$$

But as $[x]$ takes only integer values

$[x] = 1, 2$ since $[x] = 1 \Rightarrow 1 \leq x < 2$ and $[x] = 2 \Rightarrow 2 \leq x < 3$

Therefore $x \in [1, 3)$

Note:

1)

Property	$f(x)$
$f(x+y) = f(x) + f(y)$	kx
$f(x+y) = f(x)f(y)$	a^{kx}
$f(xy) = f(x)f(y)$	x^n
$f(xy) = f(x) + f(y)$	$\log x$

2) If $n(A) = m$ and $n(B) = n$ then

(a) number of functions from A and B is n^m (b) for $m \leq n$, number of one-one

functions is $\frac{n!}{(n-m)!}$

(c) for $m > n$, number of one-one functions is 0

(d) for $m \geq n$, number of onto functions are $n^m - {}^n C_1(n-1)^m + {}^n C_2(n-2)^m - {}^n C_3(n-3)^m + \dots + (-1)^{n-1} {}^n C_{n-1}$

(e) for $m < n$, number of onto functions are 0.

(f) number of constant functions is m .

3) Characteristic & Mantissa of Common Logarithm $\log_{10} x$:

As $x = [x] + \{x\}$

$\log_{10} x = [\log_{10} x] + \{\log_{10} x\}$

Where, integral part $[\log_{10} x]$ is called Characteristic & fractional part $\{\log_{10} x\}$ is called Mantissa.

Illustration : For $\log_{10} 23$,

$\log_{10} 10 < \log_{10} 23 < \log_{10} 100$

$\log_{10} 10 < \log_{10} 23 < \log_{10} 10^2$

$\log_{10} 10 < \log_{10} 23 < 2\log_{10} 10$

$1 < \log_{10} 23 < 2 \quad (\because \log_{10} 10 = 1)$

Then $[\log_{10} 23] = 1$, hence Characteristic of $\log_{10} 23$ is 1.

The characteristic of the logarithm of a number N, with 'm' digits in its integral part is 'm-1'.

Ex. 16 : Given that $\log_{10} 2 = 0.3010$, find the number of digits in the number 20^{10} .

Solution : Let $x = 20^{10}$, taking \log_{10} on either sides, we get

$$\begin{aligned} \log_{10} x &= \log_{10} (20^{10}) = 10\log_{10} 20 \\ &= 10\log_{10} (2 \times 10) = 10\{\log_{10} 2 + \log_{10} 10\} \\ &= 10\{\log_{10} 2 + 1\} = 10\{0.3010 + 1\} \\ &= 10(1.3010) = 13.010 \end{aligned}$$

That is characteristic of x is 13.

So number of digits in x is $13 + 1 = 14$

EXERCISE 6.2

- If $f(x) = 3x + 5$, $g(x) = 6x - 1$, then find
 - $(f+g)(x)$
 - $(f-g)(x)$
 - $(fg)(x)$
 - $(f/g)(x)$ and its domain.
- Let $f: \{2,4,5\} \rightarrow \{2,3,6\}$ and $g: \{2,3,6\} \rightarrow \{2,4\}$ be given by $f = \{(2,3), (4,6), (5,2)\}$ and $g = \{(2,4), (3,4), (6,2)\}$. Write down $g \circ f$
- If $f(x) = 2x^2 + 3$, $g(x) = 5x - 2$, then find
 - $f \circ g$
 - $g \circ f$
 - $f \circ f$
 - $g \circ g$
- Verify that f and g are inverse functions of each other, where
 - $f(x) = \frac{x-7}{4}$, $g(x) = 4x + 7$
 - $f(x) = x^3 + 4$, $g(x) = \sqrt[3]{x-4}$
 - $f(x) = \frac{x+3}{x-2}$, $g(x) = \frac{2x+3}{x-1}$

5) Check if the following functions have an inverse function. If yes, find the inverse function.

(a) $f(x) = 5x^2$

(b) $f(x) = 8$

(c) $f(x) = \frac{6x-7}{3}$

(d) $f(x) = \sqrt{4x+5}$

(e) $f(x) = 9x^3 + 8$

(f) $f(x) = \begin{cases} x+7 & x < 0 \\ 8-x & x \geq 0 \end{cases}$

6) If $f(x) = \begin{cases} x^2 + 3, & x \leq 2 \\ 5x + 7, & x > 2 \end{cases}$, then find

(a) $f(3)$

(b) $f(2)$

(c) $f(0)$

7) If $f(x) = \begin{cases} 4x-2, & x \leq -3 \\ 5, & -3 < x < 3 \\ x^2, & x \geq 3 \end{cases}$, then find

(a) $f(-4)$

(b) $f(-3)$

(c) $f(1)$

(d) $f(5)$

8) If $f(x) = 2|x| + 3x$, then find

(a) $f(2)$

(b) $f(-5)$

9) If $f(x) = 4[x] - 3$, where $[x]$ is greatest integer function of x , then find

(a) $f(7.2)$

(b) $f(0.5)$

(c) $f\left(-\frac{5}{2}\right)$

(d) $f(2\pi)$, where $\pi = 3.14$

10) If $f(x) = 2\{x\} + 5x$, where $\{x\}$ is fractional part function of x , then find

(a) $f(-1)$

(b) $f\left(\frac{1}{4}\right)$

(c) $f(-1.2)$

(d) $f(-6)$

11) Solve the following for x , where $|x|$ is modulus function, $[x]$ is greatest integer function, $\{x\}$ is a fractional part function.

(a) $|x+4| \geq 5$

(b) $|x-4| + |x-2| = 3$

(c) $x^2 + 7|x| + 12 = 0$

(d) $|x| \leq 3$

(e) $2|x| = 5$

(f) $[x + [x + [x]]] = 9$

(g) $\{x\} > 4$

(h) $\{x\} = 0$

(i) $\{x\} = 0.5$

(j) $2\{x\} = x + [x]$



Let's Remember

• If $f:A \rightarrow B$ is a function and $f(x) = y$, where $x \in A$ and $y \in B$, then

Domain of f is $A =$ Set of Inputs = Set of Pre-images = Set of values of x for which $y = f(x)$ is defined = Projection of graph of $f(x)$ on X-axis.

Range of f is $f(A) =$ Set of Outputs = Set of Images = Set of values of y for which $y = f(x)$ is defined = Projection of graph of $f(x)$ on Y-axis.

Co-domain of f is B .

• If $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$ then f is **one-one** and for every $y \in B$, if there exists $x \in A$ such that $f(x) = y$ then f is **onto**.

• If $f:A \rightarrow B$, $g:B \rightarrow C$ then a function $g \circ f:A \rightarrow C$ is a **composite function**.

• If $f:A \rightarrow B$, then $f^{-1}:B \rightarrow A$ is **inverse function** of f .

• If $f:\mathbb{R} \rightarrow \mathbb{R}$ is a real valued function of real variable, the following table is formed.

Type of f	Form of f	Domain of f	Range of f
Constant function	$f(x) = k$	\mathbb{R}	k
Identity function	$f(x) = x$	\mathbb{R}	\mathbb{R}
Square function	$f(x) = x^2$	\mathbb{R}	$[0, \infty)$ or \mathbb{R}^+
Cube function	$f(x) = x^3$	\mathbb{R}	\mathbb{R}
Linear function	$f(x) = ax + b$	\mathbb{R}	\mathbb{R}
Quadratic function	$f(x) = ax^2 + bx + c$	\mathbb{R}	$\left(\frac{4ac - b^2}{4a}, \infty\right)$
Cubic function	$f(x) = ax^3 + bx^2 + cx + d$	\mathbb{R}	\mathbb{R}
Square root function	$f(x) = \sqrt{x}$	$[0, \infty)$	$[0, \infty)$ or \mathbb{R}^+
Cube root function	$f(x) = \sqrt[3]{x}$	\mathbb{R}	\mathbb{R}
Rational function	$f(x) = \frac{p(x)}{q(x)}$	$\mathbb{R} - \{x \mid q(x) = 0\}$	depends on $p(x)$ and $q(x)$
Exponential function	$f(x) = a^x, a > 1$	\mathbb{R}	$(0, \infty)$
Logarithmic function	$f(x) = \log_a x, a > 1$	$(0, \infty)$ or \mathbb{R}^+	\mathbb{R}
Absolute function	$f(x) = x $	\mathbb{R}	$[0, \infty)$ or \mathbb{R}^+
Signum function	$f(x) = \text{sgn}(x)$	\mathbb{R}	$\{-1, 0, 1\}$
Greatest Integer function	$f(x) = [x]$	\mathbb{R}	\mathbb{I} (set of integers)
Fractional Part function	$f(x) = \{x\}$	\mathbb{R}	$[0, 1)$

MISCELLANEOUS EXERCISE 6

(I) Select the correct answer from given alternatives.

- If $\log(5x - 9) - \log(x + 3) = \log 2$ then $x = \dots\dots\dots$
 A) 3 B) 5 C) 2 D) 7
- If $\log_{10}(\log_{10}(\log_{10}x)) = 0$ then $x =$
 A) 1000 B) 10^{10}
 C) 10 D) 0
- Find x , if $2\log_2 x = 4$
 A) 4, -4 B) 4
 C) -4 D) not defined
- The equation $\log_{x^2} 16 + \log_{2x} 64 = 3$ has,
 A) one irrational solution
 B) no prime solution
 C) two real solutions
 D) one integral solution
- If $f(x) = \frac{1}{1-x}$, then $f(f\{f(x)\})$ is
 A) $x - 1$ B) $1 - x$ C) x D) $-x$

- 14) If $f(x) = \frac{x+3}{4x-5}$, $g(x) = \frac{3+5x}{4x-1}$ then show that $(f \circ g)(x) = x$.
- 15) Let $f: \mathbb{R} - \{2\} \rightarrow \mathbb{R}$ be defined by $f(x) = \frac{x^2-4}{x-2}$ and $g: \mathbb{R} \rightarrow \mathbb{R}$ be defined by $g(x) = x+2$. Ex whether $f = g$ or not.
- 16) Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be given by $f(x) = x+5$ for all $x \in \mathbb{R}$. Draw its graph.
- 17) Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be given by $f(x) = x^3 + 1$ for all $x \in \mathbb{R}$. Draw its graph.
- 18) For any base show that $\log(1+2+3) = \log 1 + \log 2 + \log 3$.
- 19) Find x , if $x = 3^{3 \log_3 2}$
- 20) Show that, $\log |\sqrt{x^2+1} + x| + \log |\sqrt{x^2+1} - x| = 0$
- 21) Show that, $\log \frac{a^2}{bc} + \log \frac{b^2}{ca} + \log \frac{c^2}{ab} = 0$
- 22) Simplify, $\log(\log x^4) - \log(\log x)$.
- 23) Simplify $\log_{10} \frac{28}{45} - \log_{10} \frac{35}{324} + \log_{10} \frac{325}{432} - \log_{10} \frac{13}{15}$
- 24) If $\log \left(\frac{a+b}{2} \right) = \frac{1}{2}(\log a + \log b)$, then show that $a=b$
- 25) If $b^2=ac$. prove that, $\log a + \log c = 2 \log b$
- 26) Solve for x , $\log_x(8x-3) - \log_x 4 = 2$
- 27) If $a^2 + b^2 = 7ab$, show that, $\log \left(\frac{a+b}{2} \right) = \frac{1}{2} \log a + \frac{1}{2} \log b$
- 28) If $\log \left(\frac{x-y}{5} \right) = \frac{1}{2} \log x + \frac{1}{2} \log y$, show that $x^2 + y^2 = 27xy$.
- 29) If $\log_3 [\log_2(\log_3 x)] = 1$, show that $x = 6561$.
- 30) If $f(x) = \log(1-x)$, $0 \leq x < 1$ show that $f \left(\frac{1}{1+x} \right) = f(1-x) - f(-x)$
- 31) Without using log tables, prove that $\frac{2}{5} < \log_{10} 3 < \frac{1}{2}$
- 32) Show that $7 \log \left(\frac{15}{16} \right) + 6 \log \left(\frac{8}{3} \right) + 5 \log \left(\frac{2}{5} \right) + \log \left(\frac{32}{25} \right) = \log 3$
- 33) Solve : $\sqrt{\log_2 x^4 + 4 \log_4 \sqrt{\frac{2}{x}}} = 2$
- 34) Find value of $\frac{3 + \log_{10} 343}{2 + \frac{1}{2} \log_{10} \left(\frac{49}{4} \right) + \frac{1}{2} \log_{10} \left(\frac{1}{25} \right)}$
- 35) If $\frac{\log a}{x+y-2z} = \frac{\log b}{x+y-2x} = \frac{\log c}{x+y-2y}$, show that $abc = 1$.
- 36) Show that, $\log_y x^3 \cdot \log_z y^4 \cdot \log_x z^5 = 60$
- 37) If $\frac{\log_2 a}{4} = \frac{\log_2 b}{6} = \frac{\log_2 c}{3k}$ and $a^3 b^2 c = 1$ find the value of k .
- 38) If $a^2 = b^3 = c^4 d^5$, show that $\log_a bcd = \frac{47}{30}$.
- 39) Solve the following for x , where $|x|$ is modulus function, $[x]$ is greatest interger function, $\{x\}$ is a fractional part function.
- a) $1 < |x-1| < 4$ c) $|x^2 - x - 6| = x + 2$
 c) $|x^2 - 9| + |x^2 - 4| = 5$
 d) $-2 < [x] \leq 7$ e) $2[2x-5] - 1 = 7$
 f) $[x^2] - 5[x] + 6 = 0$
 g) $[x-2] + [x+2] + \{x\} = 0$
 h) $\left[\frac{x}{2} \right] + \left[\frac{x}{3} \right] = \frac{5x}{6}$

40) Find the domain of the following functions.

a) $f(x) = \frac{x^2 + 4x + 4}{x^2 + x - 6}$

b) $f(x) = \sqrt{x-3} + \frac{1}{\log(5-x)}$

c) $f(x) = \sqrt{1 - \sqrt{1 - \sqrt{1 - x^2}}}$

d) $f(x) = x!$

e) $f(x) = {}^{5-x}P_{x-1}$

f) $f(x) = \sqrt{x-x^2} + \sqrt{5-x}$

g) $f(x) = \sqrt{\log(x^2 - 6x + 6)}$

41) Find the range of the following functions.

a) $f(x) = |x-5|$ b) $f(x) = \frac{x}{9+x^2}$

c) $f(x) = \frac{1}{1+\sqrt{x}}$ d) $f(x) = [x] - x$

e) $f(x) = 1 + 2^x + 4^x$

42) Find $(f \circ g)(x)$ and $(g \circ f)(x)$

a) $f(x) = e^x, g(x) = \log x$

b) $f(x) = \frac{x}{x+1}, g(x) = \frac{x}{1-x}$

43) Find $f(x)$ if

a) $g(x) = x^2 + x - 2$ and $(g \circ f)(x) = 4x^2 - 10x + 4$

(b) $g(x) = 1 + \sqrt{x}$ and $f[g(x)] = 3 + 2\sqrt{x} + x$.

44) Find $(f \circ f)(x)$ if

(a) $f(x) = \frac{x}{\sqrt{1+x^2}}$

(b) $f(x) = \frac{2x+1}{3x-2}$

